Pinning Down the Inner Radiative Correction in Beta Decays

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Outline

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2. Dispersive Approach
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1. The Inner Radiative Correction
The Inner Radiative Correction

- Extraction of $V_{ud}$ from beta decays:
  - (1) Superallowed beta decay
  - (2) Neutron beta decay

\[ |V_{ud}|^2 = \frac{2984.432(3) s}{\mathcal{F} t(1 + \Delta V_R)} \]

- \textit{ft values corrected by nuclear structure effects: see Misha’s talk}

\[ |V_{ud}|^2 = \frac{5099.34 s}{\tau_n(1 + 3\lambda^2)(1 + \Delta_R)} \]

\[ \Delta_R = \left( \frac{\alpha}{2\pi} \right) g(E_m) + \Delta^V_R. \]

- \textit{“nucleus-independent” correction}

- \textit{“outer” correction: sensitive to electron spectrum: see Leendert’s talk}

- \textit{“Inner radiative correction”: the part of radiative correction (RC) which is insensitive to the electron spectrum}
The Inner Radiative Correction

• Main source of uncertainty in inner RC: $\gamma_W$-box diagram

The "model-dependent" piece involves the axial component of the charged weak current:

\[ T_{\gamma W}^{\mu\nu} = \frac{1}{2} \int d^4x e^{i q \cdot x} \langle p(p) | T[J_{em}^{\mu}(x) J_{W}^{\nu}(0)] | n(p) \rangle \]

\[ = \left[ -g^{\mu\nu} + \frac{q^{\mu} q^{\nu}}{q^2} \right] T_1 + \frac{\hat{p}^{\mu} \hat{p}^{\nu}}{p \cdot q} T_2 + \frac{i \varepsilon^{\mu\nu\alpha\beta} p_{\alpha} q_{\beta}}{2p \cdot q} T_3 \]

\[ (\Delta V)_R^{VA \gamma W} = 8\pi \alpha \text{Re} \int \frac{d^4q}{(2\pi)^4} \frac{M_W^2}{M_W^2 + Q^2} \frac{Q^2 + \nu^2}{Q^4} \frac{T_3(\nu, Q^2)}{m_N \nu} \]

\[ \nu = p \cdot q / m_N \]
The Inner Radiative Correction

• Previous best determination: Marciano and Sirlin (M&S)

• Write the RC as a single-variable integral over $Q^2$, and identify the dominant physics as a function of $Q^2$.

\[
(\Delta_R^{V})_{\gamma W}^V = \frac{\alpha}{4\pi} \int_0^\infty dQ^2 \frac{M_W^2}{M_W^2 + Q^2} F(Q^2)
\]

1. Short distance: **leading OPE + perturbative QCD**
2. Intermediate distance: **VMD-inspired interpolating function + 100% uncertainty**
3. Long distance: **Elastic contribution**

Combined: \( \Delta_R^V (M & S) = 0.02361(38) \)

\[|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9994(5)\]
2. Dispersive Approach


Dispersive Approach

- \textbf{T}_3 \textbf{depends on virtual intermediate states}: theoretical modeling is less transparent
- \textbf{Dispersive treatments} to box diagrams are developed since the last ten years, relating the former to matrix elements of \textbf{on-shell intermediate states}

\[
T_{\gamma W}^{\mu \nu} = \frac{1}{2} \int d^4 x e^{i q \cdot x} \langle p(p) | T[J_{em}^\mu(x) J_W^\nu(0)] | n(p) \rangle
\]

\[
= \left[ -g^{\mu \nu} + \frac{q^{\mu} q^{\nu}}{q^2} \right] T_1 + \frac{\hat{p}^{\mu} \hat{p}^{\nu}}{p \cdot q} T_2 + \frac{i \varepsilon^{\mu \nu \alpha \beta} p_\alpha q_\beta}{2p \cdot q} T_3
\]

Hadronic tensor in inclusive scattering:

\[
W_{\gamma W}^{\mu \nu} = \frac{1}{8\pi} \int d^4 x e^{i q \cdot x} \langle p(p) | [J_{em}^\mu(x), J_W^\nu(0)] | n(p) \rangle
\]

\[
= \left[ -g^{\mu \nu} + \frac{q^{\mu} q^{\nu}}{q^2} \right] F_1 + \frac{\hat{p}^{\mu} \hat{p}^{\nu}}{p \cdot q} F_2 + \frac{i \varepsilon^{\mu \nu \alpha \beta} p_\alpha q_\beta}{2p \cdot q} F_3
\]

- We need only the contribution from the \textbf{isoscalar EM current (0)}

\[
J_{em}^\mu = J_{em}^{(I=0), \mu} + J_{em}^{(I=1), \mu}
\]
Dispersive Approach

- **Dispersion relation:**

\[
T_3^{(0)}(\omega, Q^2) = -4i\omega \int_0^1 dx \frac{F_3^{(0)}(x, Q^2)}{1 - \omega^2 x^2}
\]

\[\omega = \frac{1}{x_B} = \frac{2p \cdot q}{Q^2}\]

- **Box diagrams are expressed in terms of the “**First Nachtmann moment**” of \(F_3^{(0)}\):

\[
(\Delta V^V)_\gamma = \int_0^\infty \frac{dQ^2}{Q^2} \frac{3\alpha}{\pi} \frac{M_W^2}{M_W^2 + Q^2} M_1[F_3^{(0)}]
\]

\[M_1[F_3^{(0)}] = \int_0^1 dx \Pi(x, Q^2) F_3^N(x, Q^2)\]

\[\Pi(x, Q^2) = \frac{4}{3} \frac{1 + 2\sqrt{1 + 4m_N^2x^2/Q^2}}{(1 + \sqrt{1 + 4m_N^2x^2/Q^2})^2}\]
Dispersive Approach

- **Isospin symmetry:**
  \[
  F_3^{(0)} = -\frac{1}{4} (F_3^p - F_3^n)
  \]
  where the **flavor-diagonal structure functions** $F_3^N$ are defined through:

  \[
  W_{N}^{\mu\nu} = \frac{1}{4\pi} \int d^4x e^{i q \cdot x} \langle N(p)|[J_{em}^{\mu}(x), J_{A}^{\nu}(0)]|N(p)\rangle \\
  = \frac{i \varepsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{2 p \cdot q} F_3^N
  \]

  involving the interference between the **FULL electromagnetic current** and the **ISOVECTOR axial current**:

  \[
  J_{A}^{\mu} = \bar{u} \gamma^{\mu} \gamma_5 u - \bar{d} \gamma^{\mu} \gamma_5 d
  \]
A “phase space” diagram for $F_3^{(0)}$
Dispensive Approach

\[ F_3^{(0)} = F_{3,\text{el}}^{(0)} + F_{3,\text{inel}}^{(0)} \]

\[ F_{3,\text{inel}}^{(0)} = \begin{cases} 
F_{3,\text{DIS}}^{(0)} & Q^2 > 2\text{GeV}^2 \\
F_{3,N\pi}^{(0)} + F_{3,\text{res}}^{(0)} + F_{3,\mathbb{R}}^{(0)} & Q^2 < 2\text{GeV}^2 
\end{cases} \]

Elastic: (isoscalar) magnetic Sach FF and axial FF

\[
F_{3,\text{el}}^{(0)} = -\frac{1}{4} G_A(Q^2) G_M^{S}(Q^2) \delta(1 - x)
\]

\[ 0.0019(2) \rightarrow 0.0021(1) \]

DIS: polarized Bjorken sum rule +pQCD correction

\[
M_1[F_{3,\text{DIS}}^{(0)}] = \frac{1}{12} \left[ 1 - \tilde{C}_1 \left( \frac{\alpha S}{\pi} \right) - \tilde{C}_2 \left( \frac{\alpha S}{\pi} \right)^2 - \tilde{C}_3 \left( \frac{\alpha S}{\pi} \right)^3 + \ldots \right]
\]

\[ 0.00427 \rightarrow 0.00434 \] (mere change of integration limit)

**N_{\pi^+} Resonance: Negligible**

(Only I=1/2 intermediate states contributes)
Dispersive Approach

\[ F_{3,\text{inel}}^{(0)} = \begin{cases} 
F_{3,\text{DIS}}^{(0)} & Q^2 > 2\text{GeV}^2 \\
F_{3,N\pi}^{(0)} + F_{3,\text{res}}^{(0)} + F_{3,R}^{(0)} & Q^2 < 2\text{GeV}^2 
\end{cases} \]

Multi-hadron states: Regge model + VDM

\( (l=1)^* (l=0) \) \hspace{1cm} \( (l=1)^* (l=1) \)
Matching the 1\textsuperscript{st} Nachtmann moment of the (l=1)\textsuperscript{*}(l=1) piece to $\nu p/\bar{\nu}p$ scattering data, the (l=1)\textsuperscript{*}(l=0) piece is then deduced using Regge model+VDM.
Significant increase in the multi-hadron contribution compare to M&S result, with reduced uncertainty:

\[ 0.0003(3) \rightarrow 0.0011(2) \]
Dispersive Approach

- Reduced hadronic uncertainty in the determination of $V_{ud}$:

\[
\Delta V_R : \quad 0.02361(38) \to 0.02467(22)
\]

\[
|V_{ud}| : \quad 0.97420(21) \to 0.97370(14)
\]

\[
|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 : \quad 0.9994(5) \to 0.9984(4)
\]

(assume nothing else changes; using $V_{us}$ in PDG)

- Possible issues:
  - Quality of the neutrino data?
  - Residual model-dependence?

which leads to the discussions below.

Dispersive Approach

- Reduced hadronic uncertainty in the determination of $V_{ud}$:

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\Delta_{R}^{V} : \quad 0.02361(38) \rightarrow 0.02467(22)
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|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 : \\
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- Possible issues:
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which leads to the discussions below.
3. First-Principle Calculation

(to appear in PRL)
Recall the that we are interested in $M_1[F_3^N]$ as a function of $Q^2$. Neutrino data helps identifying dominant contributors at different $Q^2$:

$$\approx \text{elastic} + \Delta \quad Q^2 < 0.1\text{GeV}^2$$

$$M_1[F_3^N] = \begin{cases} 
\text{multi-hadron states} & 0.1\text{GeV}^2 < Q^2 < 2\text{GeV}^2 \\
\text{DIS} & Q^2 > 2\text{GeV}^2 
\end{cases}$$

Therefore, to remove the hadronic uncertainties in the box diagrams, we need to have a good handle of the first Nachtmann moment of $F_3$ at moderate $Q^2$.

Question: is there a way to calculate $M_1[F_3^N]$ from FIRST-PRINCIPLE?
First-Principle Calculation

\[ W^{\mu\nu} = \frac{1}{4\pi} \sum_X (2\pi)^4 \delta^4(p + q - px) \langle N(p) | J^{\mu}_{em} | X \rangle \langle X | J^{\nu}_A | N(p) \rangle \]

- Difficult because it involves a sum of all on-shell intermediate states.

- Recently-developed techniques in lattice calculation of PDFs (quasi-PDF, pseudo-PDF, lattice cross-section etc) do not apply because they rely on OPE that holds only at large \( Q^2 \).

- We wish to avoid direct calculations of four-point functions (noisy contractions, complicated finite-volume effect...)

A more promising approach is through the **Feynman-Hellmann theorem (FHT)**:

\[
\frac{dE_{n,\lambda}}{d\lambda} = \left\langle n_\lambda \left| \frac{\partial H_\lambda}{\partial \lambda} \right| n_\lambda \right\rangle
\]

- **Shift in energy level** → **matrix element**. Extraction of energy levels on lattice are more straightforward, avoid complicated contraction diagrams.
- **Momentum transfer** could be introduced through **periodic external potential**.
- Shows great potential in studies of:
  - Nucleon axial charge and sigma term
  - EM form factors
  - Compton amplitude
  - P-even structure functions
  - Hadron resonances
  - ......
A more promising approach is through the Feynman-Hellmann theorem (FHT):

\[ \frac{dE_{n,\lambda}}{d\lambda} = \langle n_\lambda | \frac{\partial H_{\lambda}}{\partial \lambda} | n_\lambda \rangle \]

Shift in energy level \(\to\) matrix element. Extraction of energy levels on lattice are more straightforward, avoid complicated contraction diagrams.

Nucleon Structure Functions from Operator Product Expansion on the Lattice


(QCDSF Collaboration)

- P-even structure functions
- Hadron resonances
- .......

Chambers et al., PRL 118, 242001 (2017)
First-Principle Calculation

Some warm-up:

Consider a periodic potential:

\[ V(\vec{x}) = V_0 \cos(\vec{q} \cdot \vec{x}) = \frac{1}{2} V_0 (e^{i\vec{q} \cdot \vec{x}} + e^{-i\vec{q} \cdot \vec{x}}) \]

\[ V(\vec{x}) \psi_{\vec{p}}(\vec{x}) \sim \psi_{\vec{p}+\vec{q}}(\vec{x}) + \psi_{\vec{p}-\vec{q}}(\vec{x}) \]

Off-shell condition:

\[ |\omega| < 1 \implies E(\vec{p} \pm \vec{q}) > E(\vec{p}) \]

Kinematics:

\[ q^\mu = (0, \vec{q}) \implies \omega = -\frac{2\vec{p} \cdot \vec{q}}{Q^2} \]

• Off-shell condition prohibits mixing of degenerate states through perturbation. Thus, non-degenerate perturbation theory at 1\textsuperscript{st}-order gives:

\[ \langle \vec{p} | V | \vec{p} \rangle \sim \langle \vec{p} | \vec{p} \pm \vec{q} \rangle = 0 \]

No first-order energy shift!
First-Principle Calculation

Our Strategy:

• Introduce **TWO** periodic source terms, and study the **SECOND ORDER ENERGY SHIFT**:

\[
H_\lambda = H_0 + 2\lambda_1 \int d^3x \cos(\bar{q} \cdot x)J_{em}^2(\bar{x}) - 2\lambda_2 \int d^3x \sin(\bar{q} \cdot x)J_A^3(\bar{x})
\]

\[
\left( \frac{\partial^2 E_{N,\lambda}(\vec{p})}{\partial \lambda_1 \partial \lambda_2} \right)_{\lambda=0} \equiv \frac{i q_x}{Q^2 \omega} T_3^N(\omega, Q^2).
\]


• Plugging it into the dispersion relation of \( T_3^N \):

\[
\left( \frac{\partial^2 E_{N,\lambda}(\vec{p})}{\partial \lambda_1 \partial \lambda_2} \right)_{\lambda=0} = \frac{4 q_x}{Q^2} \int_0^1 dx \frac{F_3^N(x, Q^2)}{1 - \omega^2 x^2},
\]

Central result!!!

2\textsuperscript{nd} order Energy shift \quad \text{FHT} \quad \text{Generalized Forward Compton tensor} \quad \text{DR} \quad \text{Structure Function}
First Principle Calculation

Isolating the inelastic contribution:

First Nachtmann moment:

\[ M_1[F_3^N] = M_1[F_3^N]_{el} + \int_0^{x_\pi} dx \Pi(x, Q^2) F_3^N(x, Q^2) \]

Energy shift:

\[ \frac{Q^2}{4q_x} \left( \frac{\partial^2 E_{N,\lambda}}{\partial \lambda_1 \partial \lambda_2} \right)_{\lambda=0} = \frac{Q^2}{4q_x} \left( \frac{\partial^2 E_{N,\lambda}}{\partial \lambda_1 \partial \lambda_2} \right)_{\lambda=0}^{el} + \int_0^{x_\pi} dx \Lambda(x, \omega) F_3^N(x, Q^2) \]

Elastic piece fully described by form factors (experiment + lattice):

\[ F_{3,el}^N = -2I_3^N G_A(Q^2) G_M^N(Q^2) \delta(1-x) \]

Inelastic contribution starts from the pion production threshold:

\[ x_\pi = \frac{Q^2}{2m_N M_\pi + M_\pi^2 + Q^2} \]

Small at small \( Q^2 \)!
First-Principle Calculation

- Lattice momenta are **discrete**:
  \[
  \vec{p} = \frac{2\pi}{L} (n_{px}, n_{py}, n_{pz}), \quad \vec{q} = \frac{2\pi}{L} (n_{qx}, n_{qy}, n_{qz})
  \]

- Requiring \(Q^2\) at the hadronic scale and the off-shell condition imply:
  \[
  \frac{4\pi^2}{L^2} (n_{qx}^2 + n_{qy}^2 + n_{qz}^2) \lesssim 1 \text{ GeV}^2 \\
  \frac{2|n_{px}n_{qx} + n_{py}n_{qy} + n_{pz}n_{qz}|}{n_{qx}^2 + n_{qy}^2 + n_{qz}^2} < 1.
  \]

- **A concrete example**: 
  \[
  L \approx 4.47 \text{fm} \quad \vec{q} = \frac{2\pi}{L} (2, 1, 0)
  \]
  impose the restriction: 
  \[Q^2 \approx 0.38 \text{GeV}^2\]

  Allowed values for \(\omega\):
  \[
  |\omega| = 0, \quad \frac{2}{5}, \quad \frac{4}{5}
  \]

  Pion production threshold:
  \[x_\pi \approx 0.58\]
  (assume physical pion mass)
Reconstructing the first Nachtmann moment from energy shifts

\[ M_1[F_3^N]_{\text{inel}} = \int_0^{x_{\pi}} dx \Pi(x, Q^2) F_3^N(x, Q^2) \]

\[ = \frac{Q^2}{4q_x} \left( \frac{\partial^2 E_{N,\lambda}}{\partial \lambda_1 \partial \lambda_2} \right)_{\lambda=0}^{\text{inel}} \]

\[ \Lambda(x, \omega) = (1 - \omega^2 x^2)^{-1} \]

\[ \Pi(x, Q^2) \approx a \Lambda(x, 0) + b \Lambda(x, 2/5) + c \Lambda(x, 4/5) \]

\[ a = 19.3273 \]
\[ b = -21.9879 \]
\[ c = 3.65565 \]

A very good reconstruction!
First-Principle Calculation

Reconstructing the first Nachtmann moment from energy shifts

\[ M_1[F_3^N]_{\text{inel}} = \int_0^{x\pi} dx \Pi(x, Q^2) F_3^N (x, Q^2) \]

\[ \frac{Q^2}{4q_x} \left( \frac{\partial^2 E_{N,\lambda}}{\partial \lambda_1 \partial \lambda_2} \right)_{\lambda=0} = \int_0^{x\pi} dx \Lambda(x, \omega) F_3^N (x, Q^2) \]

\[ \Lambda(x, \omega) = (1 - \omega^2 x^2)^{-1} \]

\[ \Pi(x, Q^2) \approx a \Lambda(x, 0) + b \Lambda(x, 2/5) + c \Lambda(x, 4/5) \]

\[ M_1[F_3^N]_{\text{inel}} \approx \frac{Q^2}{4q_x} \left[ a \left( \frac{\partial^2 E_{N,\lambda}}{\partial \lambda_1 \partial \lambda_2} \right)_{\lambda=0}^{\omega=0} + b \left( \frac{\partial^2 E_{N,\lambda}}{\partial \lambda_1 \partial \lambda_2} \right)_{\lambda=0}^{\omega=2/5} + c \left( \frac{\partial^2 E_{N,\lambda}}{\partial \lambda_1 \partial \lambda_2} \right)_{\lambda=0}^{\omega=4/5} \right]_{\text{inel}} \]

\[ a = 19.3273 \]
\[ b = -21.9879 \]
\[ c = 3.65565 \]
First-Principle Calculation

Reconstructing the first Nachtmann moment from energy shifts

\[ M_1[F_3^N]_{\text{inel}} = \int_0^{x_\pi} dx \Pi(x, Q^2) F_3^N(x, Q^2) \left( \frac{Q^2}{4q_x} \left( \frac{\partial^2 E_{N,\lambda}}{\partial \lambda_1 \partial \lambda_2} \right)_{\lambda=0} \right) = \int_0^{x_\pi} dx \Lambda(x, \omega) F_3^N(x, Q^2) \]

- What about smaller \( Q^2 \)?
  - The available values of \( \omega \) is less so the reconstruction of \( \Lambda(x, Q^2) \) will be less satisfactory.
  - However, \( \omega=0 \) (zero proton momentum) is ALWAYS an accessible point; that gives the first **Mellin** moment of \( F_3^N \), which sets important constraints on its model parameterization.
1. The axial $\gamma W$ box diagram is one of the main sources of theoretical uncertainty in the extraction of $V_{ud}$ through neutron and superallowed beta decay.

2. The application of dispersion relation utilizing $\nu p/\bar{\nu}p$ scattering data reduces the uncertainty in the $\gamma W$ box diagram by a half, but at the same time raises tension with the first row CKM unitarity.

3. A recent proposal that hybridizes the dispersion relation and computations of shifted energy levels on lattice may, for the first time, lead to a first-principle theoretical calculation of the $\gamma W$ box.