Overview/Status of the Bottle Method

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Measurement Principle

Storage of UCNs (or VCNs) by means of

- the mean Fermi potential $2\pi\hbar^2 Na/m$
- the magnetic interaction $-\mu \cdot B$
- (often) plus vertical confinement by gravity: *mgh*

Principle of Measurement Cycle

- Load UCNs (VCNs) into the trap
- Store for (at least) two periods: Δt_1 (short) and Δt_2 (long, $\sim \tau_n$); $\Delta t = \Delta t_2 - \Delta t_1$
- Count survivors: N_1, N_2
- If β -decay is the only loss process:

$$\tau_n = \frac{\Delta t}{\ln(N_1 / N_2)}$$

• End of simple picture: Now the real physics (In short: All τ_n experiments are "tour de force".)

Complications, first stage

• Additional losses for material bottles:

$$\tau_{n}^{-1} \rightarrow \tau^{-1} = \tau_{n}^{-1} + \tau_{inel}^{-1} + \tau_{cap}^{-1} + \tau_{leaks}^{-1} + \tau_{gas}^{-1} + \tau_{q-el}^{-1} + \tau_{q-stable}^{-1} + \tau_{vibr}^{-1} + \dots$$

q-el: quasi-elastic, e.g., due to visco-elastic surface waves in liquids; "small heating" (?) **q-stable**: long-lived untrapped

• Additional losses for magnetic bottles: $\tau^{-1} = \tau_n^{-1} + \tau_{sf}^{-1} + \tau_{leaks}^{-1} + \tau_{gas}^{-1} + \tau_{q-stable}^{-1} + \tau_{vibr}^{-1} + \tau_{AC}^{-1} + (\text{not detected}) + \dots$

leaks: e.g., due to weak field near microcracks in NdFeB permanent magnets **not detected**: UCNs in quasi-stable orbits that never make it to the detector

sf: Majorana spin flip extended to trapped particles (Walstrom 2009, Steyerl 2012); AC: AC noise of electromagnets;

- Most loss rates (e.g., τ_{inel}^{-1} , τ_{cap}^{-1}) are not constant throughout a measurement cycle. UCN spectra soften progressively since wall reflection frequency and loss/reflection increase with energy.
- Fomblin cross sections measured by VCN transmission are only guides; agreement with stored UCN loss rate within a factor 1.5 was considered very good in MAMBO I experiment.
- Initial spectra are not well known.
- Spectra change also due to quasi-elastic scattering; in some cases (MAMBO I) all transitions, UCN→VCN, VCN→VCN, VCN→UCN, UCN→UCN and both up and down scattering must be taken into account. Example MAMBO I and MAMBO II
- Residual gas loss is difficult to quantify. Mass spectrometric analysis of the typical "dirty" gas is not trivial and precise direct measurement with deliberately deteriorated vacuum takes too much time.

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Schematic view of MAMBO I Mampe *et al.* (1989)



Features:

Rectangular glass box; Renewable Fomblin coating; Movable rear wall with sinusoidal undulations; UCN and VCN admitted; Use scaling.

Extrapolation technique used for MAMBO I



Strategies suitable to cope with aspects of spectral change during a cycle: **Scaling**

- Scaling (Pendlebury, Mampe, Ageron) for MAMBO I and MAMBO II: In a measurement cycle make all filling, storage and emptying intervals $(\Delta t_{\rm f}, \Delta t_1, \Delta t_2, ... \Delta t_{\rm e})$ proportional to $\lambda = 4V/S$;
- For an up-down symmetric trap λ is a good measure even in the presence of gravity as long as all UCN have enough energy to reach the roof;
- With scaling, the net loss is the same in large and small traps;
- \Rightarrow the spectra develop in the same way and measured values τ_{st}^{-1} become comparable.
- This is not exact in the presence of quasi-elastic scattering (even if VCN are excluded by a pre-storage chamber); quasi-elastic cooling below the "roof energy" is not restricted.
- From experience with simulations: In practice scaling works very well.

First analysis (1989) neglected quasi-elastic scattering on viscoelastic surface waves $\Rightarrow \tau_n = 887.6\pm 3 \text{ s};$ These were later (2010) taken into account $\Rightarrow \tau_n = 882.5\pm 2.1 \text{ s}$



For isotropic incidence:

Increase of mean wall loss coefficient $\langle \mu(k_i) \rangle$ over loss $\langle \mu_i(k_i) \rangle$ without q-elastic scattering; \Rightarrow significant effect extends deep down into UCN region

Spectral change in MAMBO I simulated



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Schematic of Mambo II; Pichlmaier *et al.* (2000) and (2010)



New elements:

spectral cleaning in prestorage chamber;

monitoring of residual gas;

$$\tau_{\rm n} = 880.7 \pm 1.3_{\rm stat} \pm 1.2_{\rm sys} \, {\rm s}$$

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- Residual gas loss is difficult to quantify. Mass spectrometric analysis of the typical "dirty" gas is not trivial and precise direct measurement with intentionally deteriorated vacuum takes too much time. MAMBO II made a serious effort.
- Like ³He in ⁴He, residual gas in a "dirty vacuum" may be a significant problem.

Complications, third stage

- Quasi-stable orbits (or marginally trapped or persistently untrapped UCNs).
- Their energy is somewhat above barrier but in magnetic fields and material bottles like cylinders or rectangular boxes with smooth, flat walls these orbits may persist a long time of order τ_n .
- Spectral cleaning takes long and may be inefficient.
- In the Paul experiment (magnetic VCN storage 1989) pure n beta-decay was reached only after ~300 s (hopefully).

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Paul et al. (1989); NESTOR neutron storage ring



Fig. 4. Coil configuration of the magnetic neutron storage ring NESTOR and influence of the gravitational potential on the beam position.

Features:

Magnetic hexapole with decapole component; Uses VCN up to ~20 m/s;

Beam scrapers for severe phase space limitation to suppress betatron oscillations;

Conservative error estimate.

NESTOR data; Paul et al. (1989)



Figure 17. Logarithmic decrease of the number of stored neutrons with time.

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- Spectral cleaning takes long and may be inefficient.
- In the Paul experiment (magnetic VCN storage 1989) pure n beta-decay was reached only after ~10³ s (hopefully).
- In Gravitrap: UCN in quasi-stable orbits may be counted in a "wrong" energy group or not counted at all.

Gravitational UCN storage system using Low Temperature Fomblin; Serebrov *et al.* (2005)



- 1 neutron guide from UCN turbine;
- 8 cylindrical UCN storage traps;
- 9 large conical channel;
- 12 UCN detector;
- 14 evaporator for deposition of frozen (glassy) LTF; LTF = low-temp. Fomblin
- 1-8-12 entrance/exit channel
- Five angular positions of trap aperture define five energy intervals with calculated values γ
- $loss/s = \gamma \eta$
- $\tau_n = 878.5 {\pm} 0.7_{stat} {\pm} 0.3_{stat} s$

Rotating cvlindrical vessel of Gravitrap



Х

Typical measuring cycle



- filling 160 s (time of trap rotation (35 s) to monitoring position is included);
- 2. monitoring 300 s;
- holding 300 s or 2000 s (time of trap rotation (7 s) to holding position is included);
- emptying has 5 periods 150 s, 100 s, 100 s, 100 s, 150 s (time of trap rotation (2.3 s, 2.3 s, 2.3 s, 3.5 s, 24.5 s) to each position is included);
- measurement of background 100 s.

$$N(t_2) = N(t_1) \cdot \exp\left(-\frac{t - t_1}{\tau_{st}}\right)$$

$$\tau_{st} = \frac{t_2 - t_1}{\ln\left(N(t_1)/N(t_2)\right)}$$

Fourth stage: surface roughness

- Micro-roughness may be characterized by mean height *h* and lateral correlation length *w*;
- "dense roughness (jagged)": *h≥w* as for the alps;
- "soft roughness or macroscopic waviness": h «w as for rolling hills or for glass;
- Reflected beam: partly specular $[I_{spec}$ with probability $1-\xi(\theta_i)] + partly diffuse <math>[I_{dif}$ with probability $\xi(\theta_i)]$;
- "dense roughness": I_{dif} is isotropic; i.e., $I_{dif}(\theta_i, \theta) \propto (\cos\theta_i \cos\theta) \cos\theta$ (asymmetric Lambert factor $\cos\theta$ needed for detailed balance);
- $I_{dif}(\theta) \propto \text{const.} \times \cos\theta$ is a poor approximation missing the fact $I_{dif} \Rightarrow 0$ for glancing incidence ("sliding UCNs")
- "soft roughness": I_{dif} is limited to angular range $\sim h/w \ll l$ about specular reflection:

 $I_{\rm dif}(\Omega, \Omega_i) \propto (\text{symmetric function of } \Omega_i \text{ and } \Omega) \times \cos\theta$

incident angle: Ω_i ; θ_i , φ_i ; scattered angle: Ω ; θ , φ ; θ is measured from the surface normal

Surface roughness \Leftrightarrow Quasi-stable orbits

- Symmetric trap shape (like ○, □) in combination with soft roughness ⇒ slow diffusion in phase space;
- After *N* reflections: Δθ < (h/w) N^{1/2} away from specular;
 (less sign because the specular part of reflectivity does not contribute)
- For $h/w \approx 10^{-2}$, $N \approx 10^{2}$: $\Delta \theta < 0.1 \ll 1$;
- "Persistently untrapped" particles may be counted
 - after the short storage time Δt_1 or
 - after a long storage time Δt_2 or
 - during a storage time when, supposedly, background is measured;
 - In Gravitrap: at times corresponding to a different energy group
 ⇒ mixing of energy groups 1 to 5;
 - In all cases: at the wrong time.

The case of purely of almost specular reflection (SR)

• SR leads to highly stable orbits in cylindrical trap.



UCN "sliding" along the cylindrical rim and being reflected from vertical side walls

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Effects of quasi-stable orbits

- In extrapolations to zero wall loss $(\tau_{st}^{-1} \text{ vs. } x)$, *x* is the inverse mean free path $(\lambda^{-1} = S/4V)$ (in MAMBO). In the Gravitrap, *x* is the quantity γ .
- γ must be calculated by simulations for each spectral interval 1 to 5;
- The result depends strongly on the roughness model used;
- \Rightarrow The *x*-axis of extrapolation plots is uncertain;
- The τ_{st}^{-1} values (on the *y*-axis) may also be uncertain because
 - due to spectral change the detector efficiency is different for counting N_1 and N_2 ; estimate for Gravitrap: $\Delta \varepsilon / \varepsilon \approx 4\%$ for $\Delta h/h = 1\%$. (*h* is the mean energy.)
 - unnoticed loss of quasi-stable UCN during long storage \Rightarrow too few counts N_2 relative to N_1 ;
 - \Rightarrow calculated τ_n tends to be too low.



Other effects of quasi-stable orbits

- In extrapolations to zero wall loss $(\tau_{st}^{-1} \text{ vs. } x)$, *x* may be the inverse mean free path $(\lambda^{-1} = S/4V)$ (in MAMBO) or a calculated quantity like γ (Gravitrap).
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Other effects of quasi-stable orbits

• Not recommended: The "easy way" out by insisting on the simplest roughness model

$$I_{\text{refl}} = (1-\xi) I_{\text{spec}} + \xi I_{\text{dif}}$$

with perfectly diffuse I_{dif} and ξ independent of incident angle θ_i .

- Why? Because presumed scattering into large angles may pretend that there are no quasi-stable orbits, even for fairly small ξ .
- Not recommended either: not to talk about *very* small ξ . For $\xi=0$ (purely specular reflection) quasi-stable orbits are unavoidable.
- For the glassy surface of frozen low-temperature Fomblin in the Gravitrap system: we expect the roughness to be "soft", not "jagged";
- Thus quasi-stable orbits are unavoidable.

Strategies suitable of coping with aspects of spectral change during a cycle: Monitoring upscattered n's

- In experiments of Arzumanov, Bondarenko, Morozov, Mampe: Detect the up-scattered neutrons escaping the UCN trap by an array of thermal neutron counters;
- They use scaling, two geometries *A* and *B* (internal and external volume or with/without additional surface) and, both UCN counts and thermal neutron counts;
- This provides an independent way of measuring the wall loss;
- ⇒ In principle, the neutron lifetime can be determined without the need of linear extrapolation to zero loss.

Arzumanov *et al.* "double bottle" (2014) with measurement of up-scattered UCNs



Fig.1. A scheme of the experimental set-up for the neutron lifetime measurement. 1 -the entrance neutron guide, 2 -the UCN source shutter, 3 -the input shutter, 4 -fluid fluorine polymer, 5 -the copper cylinder, 6 -the cooling coil, 7 -the polyethylene disk, 8 -thermal neutron counters, 9 -the pumping tube, 10 -the cooler tube, 11 -the valve of the He filling line, 12 -the tube of the high-vacuum line, 13 -the vacuum set-up chamber, 14 - copper stripes, 15 - the additional surface above the trap bottom and the entrance shutter, 16 - the entrance plane shutter, 17 -the pumping tube for the chamber bottom, 18 - the detector shutter, 19 -the UCN detector, 20 -a horizontal cross section of the set-up with blocks of polyethylene reflector for thermal neutrons.

With a similar system: Mampe *et al.* (1993) $\tau_n = 882.6 \pm 2.7 \text{ s};$

Arzumanov *et al.* (2000) $\tau_n = 885.4 \pm 0.9_{stat} \pm 0.4_{sys} s;$ Corrected (2012): $\tau_n = 881.6 \pm 0.8_{stat} \pm 1.9_{sys} s;$

[New experiment (2014), Preliminary value: $\tau_n = 880.2 \pm 1.2 \text{ s;}$]

Arzumanov et al. (2000)



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Strategies suitable of coping with aspects of spectral change during a cycle: **Monitoring upscattered n's**

- In practice: corrections required for
 - UCN detector dead times
 - differences in UCN detection efficiencies for geometries A and B and for short vs. long storage
 - differences in thermal n detection efficiencies for geometries A and B (measured using a flexible guide tube)
 - residual gas;
 - temperature difference between geometries *A* and *B*;
 - "weak heating" (?)

Magnetic bottle; Ezhov et al. (2009)



One spin state confined by

- 20-pole permanent magnets magnetized
 - $\dots \rightarrow \leftarrow \rightarrow \leftarrow \dots$ around periphery;
- solenoid at the bottom;
- gravity on top;
- loading from the top by lowering a lift;
- outer solenoid provides bias field eliminating zero points
- or to force depolarization as a systematic check;
- Further checks by counting the spin-flipped UCN

First result: $\tau_n = 878.2 \pm 1.9$ s

NIST UCN lifetime apparatus



Apparent difficulties with the required ultra-pure ⁴He: so far (2006): $\tau_n = 831(+58,-51)$ s

LANL magneto-gravitational "bowl" (Walstrom *et al.* 2009, Young *et al.* 2014)



LANL magneto-gravitational "bowl": first data



Ideal 2D Halbach magnetic field (Model of LANL "bowl" system)

Current of spin-flipped UCN; Loss is determined by leakage at lower and upper endpoint

Conclusions from the analysis of Walstrom *et al.* (2009) and Steyerl *et al.* (2012):

- Majorana's (1932) estimate of spin-flip probability for free spins moving through a rotating field is not applicable to UCNs confined in space.
- In realistic magnetic trapping fields (like that of the LANL "bowl") the loss is much larger than the Majorana prediction [exp(-10⁶)].
- For a vertical drop in the LANL "bowl" the loss per bounce is of order 10⁻²² per bounce in the field.
- Allowing for sideways motion in the plane of the Halbach field the loss per bounce can be larger by ~10 orders of magnitude. field.
- But it should still be low enough to allow a lifetime measurement with precision 10⁻⁴.

Conclusions from the analysis of Walstrom *et al.* (2009) and Steyerl *et al.* (2012):

• On the other hand, the correction to τ_n due to depolarization in Ezhov's experiment apparently was of order 0.5% (not <10⁻⁴ as expected).

Other current n lifetime projects using magnetic UCN

- HOPE (ILL, Zimmer, Leung 2009): Octupole trapping field generated by 32 NdFeB permanent magnets on cylindrical surface; closed on top and bottom by solenoids; detection: by counting surviving UCNs and/or e⁻ or p detection.
- PENeLOPE (Munich, Materne *et al.* 2009): Vertical superconducting quadrupole; large volume of ~700. detection.
- Lifetime project at Mainz Triga reactor...
- All aim at a precision $\sim 10^{-4}$ for τ_n .

"Global picture" of τ_n bottle experiments (my understanding 2014)

- UCN experiments, almost unavoidably, are done in "dirty" vacuum environments (high-vacuum baking usually impossible due to presence of detectors at r.t., sensitive guide surfaces, moving parts, large and complex volumes);
- Residual gas composition is often not well known; even if known, cross sections are often "educated guesses";
- Uncertainties are easily underestimated;
- In the PDG 2014 summary of 7 τ_n values one (Gravitrap 2005) carries ~60% of the total statistical weight.
- The remaining four bottle experiments give $\langle \tau_n \rangle \cong 882(1)$ s.
- The difference between beam and bottle experiments would be $\Delta < \tau_n > = 888(2) 882(1) \text{ s} = 6(2.3) \text{ s which is } 2.6 \text{ s}.$

"Global picture" of τ_n bottle experiments (my understanding 2014)

• Nesvizhevsky's "exotic idea" (2013): Do UCNs experience additional loss by scattering on a two-dimensional levitating "cloud" of nanoparticles interacting with the surface via van der Waals/Casimir-Polder forces? Both on liquid and solid suerfaces.

"Lifetime Conclusion" of Boris Yerozolimsky:

"If you try to improve the τ_n value to the level ~10⁻³ or better you will run against a brick wall of exponentially growing problems."

