Chiral Perturbation Theory for $\eta \to 3\pi$ and $\eta \to \pi^0\gamma\gamma$

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Hadronic Probes of Fundamental Symmetries, Amherst, 6-8 March 2014

ChPT aspects of $\eta \to 3\pi$ and $\eta \to \pi^0\gamma\gamma$

1. Older reviews
2. $\eta \to 3\pi$: Some model independent comments/results
   - Definitions
   - Experiment
   - Why?
3. ChPT
4. $\eta \to 3\pi$ in ChPT
   - LO
   - LO and NLO
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5. $\eta \to \pi^0\gamma\gamma$
   - p^4
   - Experiment
   - Loops at p^6, p^8
   - All else
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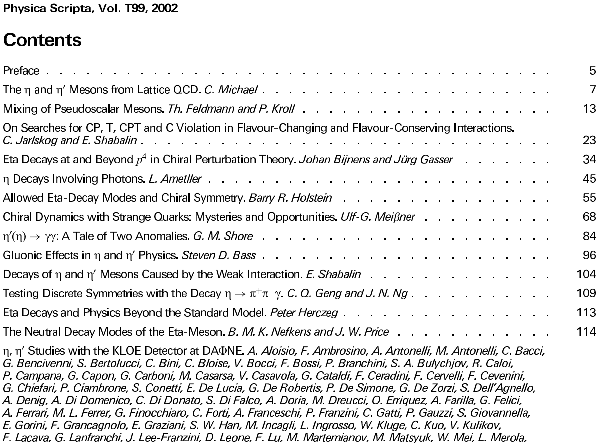
6. Conclusions

Eta Physics Handbook: ETA01



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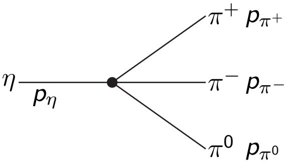
Definitions: $\eta \to 3\pi$

Reviews: JB, Gasser, Phys.Scripta T99(2002)34 [hep-ph/0202242]
JB, Acta Phys. Slov. 56(2005)305 [hep-ph/0511076]



$$\begin{aligned} s &= (p_{\pi^+}+p_{\pi^-})^2 = (p_\eta - p_{\pi^0})^2 \\ t &= (p_{\pi^-}+p_{\pi^0})^2 = (p_\eta - p_{\pi^+})^2 \\ u &= (p_{\pi^+}+p_{\pi^0})^2 = (p_\eta - p_{\pi^-})^2 \end{aligned}$$

$$s+t+u = m_\eta^2 + 2m_{\pi^+}^2 + m_{\pi^0}^2 \equiv 3s_0\,.$$

$$\langle \pi^0\pi^+\pi^-\text{out}|\eta\rangle = i\,(2\pi)^4\,\delta^4\left(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}\right)\,A(s,t,u)\,.$$

$$\langle \pi^0\pi^0\pi^0\text{out}|\eta\rangle = i\,(2\pi)^4\;\delta^4\left(p_\eta - p_1 - p_2 - p_3\right)\,\overline{A}(s_1,s_2,s_3)$$

$$\overline{A}(s_1,s_2,s_3) = A(s_1,s_2,s_3) + A(s_2,s_3,s_1) + A(s_3,s_1,s_2)$$

Obervables: $\Gamma(\eta\to\pi^+\pi^-\pi^0)$ and $r = \frac{\Gamma(\eta\to\pi^0\pi^0\pi^0)}{\Gamma(\eta\to\pi^+\pi^-\pi^0)}$

Definitions: Dalitz plot

$$\begin{aligned} x &= \sqrt{3}\frac{T_+ - T_-}{Q_\eta} = \frac{\sqrt{3}}{2m_\eta Q_\eta}(u-t) \\ y &= \frac{3T_0}{Q_\eta} - 1 = \frac{3\left((m_\eta - m_{\pi^o})^2 - s\right)}{2m_\eta Q_\eta} - 1 \overset{\text{iso}}{=} \frac{3}{2m_\eta Q_\eta}(s_0 - s) \\ Q_\eta &= m_\eta - 2m_{\pi^+} - m_{\pi^0} \end{aligned}$$

T^i is the kinetic energy of pion π^i

$$z = \frac{2}{3}\sum_{i=1,3}\left(\frac{3E_i - m_\eta}{m_\eta - 3m_{\pi^0}}\right)^2$$ E_i is the energy of pion π^i

$$\begin{aligned} |M|^2 &= A_0^2\left(1 + ay + by^2 + dx^2 + fy^3 + gx^2y + \cdots\right) \\ |\overline{M}|^2 &= \overline{A}_0^2\left(1 + 2\alpha z + \cdots\right) \end{aligned}$$

Note: neutral, next order: x and y appear separately

Relations

Expand amplitudes and use isospin: JB, Ghorbani, arXiv:0709.0230

$$M(s,t,u) = A\left(1 + \tilde{a}(s-s_0) + \tilde{b}(s-s_0)^2 + \tilde{d}(u-t)^2 + \cdots\right)$$

$$\overline{M}(s,t,u) = A\left(3 + (\tilde{b} + 3\tilde{d})\left((s-s_0)^2 + (t-s_0)^2 + (u-s_0)^2\right)\right)$$

Gives relations ($R_\eta = (2m_\eta Q_\eta)/3$)

$$a = -2R_\eta \,\mathrm{Re}(\tilde{a})\,, \quad b = R_\eta^2\left(|\tilde{a}|^2 + 2\mathrm{Re}(\tilde{b})\right)\,, \quad d = 6R_\eta^2\,\mathrm{Re}(\tilde{d})\,.$$

$$\alpha = \frac{1}{2}R_\eta^2\,\mathrm{Re}\left(\tilde{b} + 3\tilde{d}\right) = \frac{1}{4}\left(d + b - R_\eta^2|\tilde{a}|^2\right) \leq \frac{1}{4}\left(d + b - \frac{1}{4}a^2\right)$$

equality if $\mathrm{Im}(\tilde{a}) = 0$

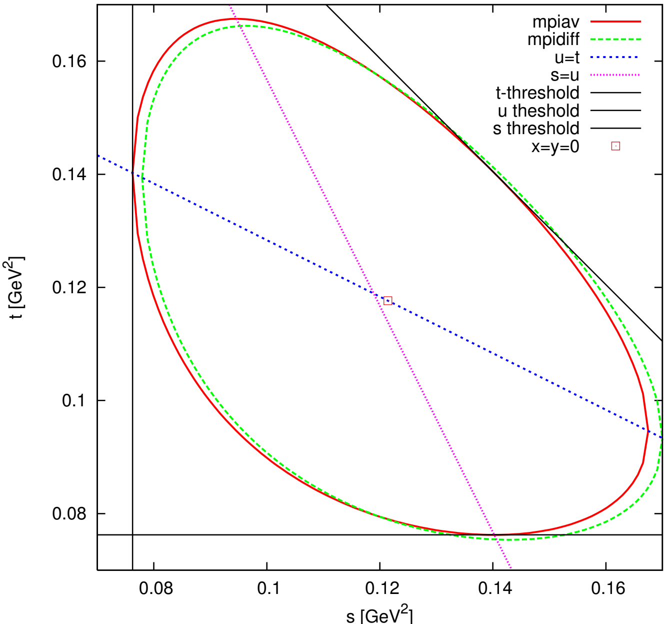
Relations

Consequences:

- Relations between the charged and neutral decay
- Relations between r and Dalitz plot
  (see also Gasser, Leutwyler, Nucl. Phys. B **250** (1985) 539)
- If you can calculate $\mathrm{Im}(\tilde{a})$ then relation:
  nonrelativistic pion EFT

  Schneider, Kubis and Ditsche, JHEP **1102** (2011) 028 [1010.3946].

Definitions: Dalitz plot



x variation: vertical

y variation: parallel to $t = u$

Experiment: Decay rates

Width: determined from $\Gamma(\eta \to \gamma\gamma)$ and Branching ratios
Using the PDG12 partial update 2013 numbers

$\Gamma(\eta \to \pi^+\pi^-\pi^0) = 300 \pm 12$ eV (in JB,Ghorbani 295 ± 17 eV)

r: 1.426 ± 0.026 (our fit)
1.48 ± 0.05 (our average)

Experiment: charged

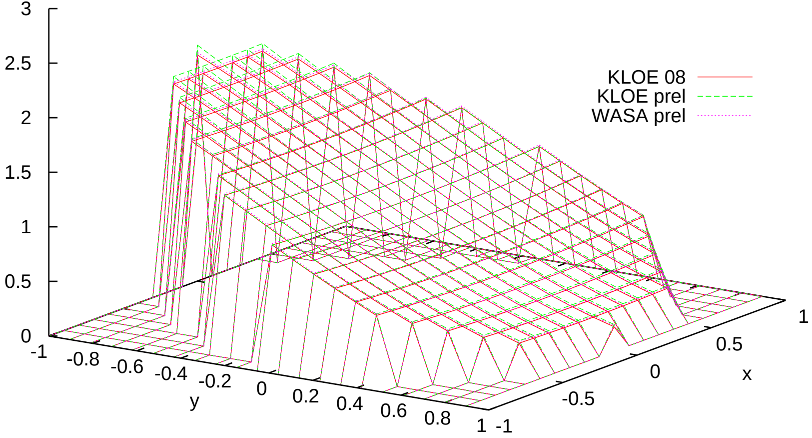
Exp.	a	b	d	f
WASA (prel)	$-1.104(3)$	$0.144(3)$	$0.073(3)$	$0.153(6)$
KLOE (prel)	$-1.074(23)(3)$	$0.179(27)(8)$	$0.059(25)(10)$	$0.089(58)$
KLOE	$-1.090(5)(^{+8}_{-19})$	$0.124(6)(10)$	$0.057(6)(^{+7}_{-16})$	$0.14(1)(2)$
Crystal Barrel	$-1.22(7)$	$0.22(11)$	$0.06(4)$ (input)	
Layter et al.	$-1.08(14)$	$0.034(27)$	$0.046(31)$	
Gormley et al.	$-1.17(2)(21)$	$0.21(3)$	$0.06(4)$	

Crystal Barrel: d input, but a and b insensitive to d

Large correlations: KLOE:
$$\begin{array}{ccccc} & a & b & d & f \\ a & 1 & -0.226 & -0.405 & -0.795 \\ b & & 1 & 0.358 & 0.261 \\ d & & & 1 & 0.113 \\ f & & & & 1 \end{array}$$

Experiment: charged

But very good agreement:



Experiment: neutral

See talks by Kupsc and Unverzagt

Exp.	α
GAMS2000	-0.022 ± 0.023
SND	$-0.010 \pm 0.021 \pm 0.010$
Crystal Barrel	$-0.052 \pm 0.017 \pm 0.010$
Crystal Ball (BNL)	-0.031 ± 0.004
WASA/CELSIUS	$-0.026 \pm 0.010 \pm 0.010$
KLOE	$-0.0301 \pm 0.0035^{+0.0022}_{-0.0035}$
WASA@COSY	$-0.027 \pm 0.008 \pm 0.005$
Crystal Ball (MAMI-B)	$-0.032 \pm 0.002 \pm 0.002$
Crystal Ball (MAMI-C)	-0.032 ± 0.003

All experiments in good agreement

Why is $\eta \to 3\pi$ interesting?

- Pions are in $I = 1$ state $\Longrightarrow A \sim (m_u - m_d)$ or α_{em}
- α_{em} effect is small
    - but is there via $(m_{\pi^+} - m_{\pi^0})$ in kinematics
    - Lowest order vanishes (current algebra)
    - $\alpha \hat{m}$ and αm_s small

      Baur, Kambor, Wyler, Nucl. Phys. B **460** (1996) 127
    - $\eta \to \pi^+\pi^-\pi^0\gamma$ needs to be included directly

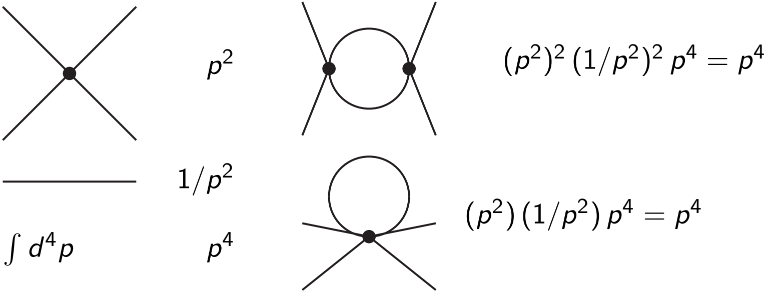
      Ditsche, Kubis, Meissner, Eur. Phys. J. C **60** (2009) 83 [0812.0344]

      Estimates the corrections of $\alpha(m_u - m_d)$ as well
    - Conclusion: at the precision I will discuss not relevant
    - Exception: Cusps and Coulomb at $\pi^+\pi^-$ thresholds
- So $\eta \to 3\pi$ gives a handle on $m_u - m_d$

Chiral Perturbation Theory

Degrees of freedom: Goldstone Bosons from Chiral Symmetry Spontaneous Breakdown (without η')
Power counting: Dimensional counting in momenta/masses
Expected breakdown scale: Resonances, so M_ρ or higher depending on the channel

Power counting in momenta: Meson loops



p^2

$1/p^2$

$\int d^4p$ $\quad p^4$

$(p^2)^2\,(1/p^2)^2\,p^4 = p^4$

$(p^2)\,(1/p^2)\,p^4 = p^4$

Lagrangians

$U(\phi) = \exp(i\sqrt{2}\Phi/F_0)$ parametrizes Goldstone Bosons

$$\Phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\,\eta_8}{\sqrt{6}} \end{pmatrix} .$$

LO Lagrangian: $\mathcal{L}_2 = \frac{F_0^2}{4}\{\langle D_\mu U^\dagger D^\mu U\rangle + \langle \chi^\dagger U + \chi U^\dagger\rangle\}\,,$

$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu\,,$
left and right external currents: $r(l)_\mu = v_\mu + (-) a_\mu$

Scalar and pseudoscalar external densities: $\chi = 2B_0(s + ip)$
quark masses via scalar density: $s = \mathcal{M} + \cdots$

$\langle A\rangle = Tr_F\,(A)$

Lagrangians

$$\begin{aligned}
\mathcal{L}_4 &= L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\
&+L_3\langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U\rangle + L_4\langle D^\mu U^\dagger D_\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
&+L_5\langle D^\mu U^\dagger D_\mu U(\chi^\dagger U + U^\dagger\chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
&+L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
&-iL_9\langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U\rangle \\
&+L_{10}\langle U^\dagger F^R_{\mu\nu} U F^{L\mu\nu}\rangle + H_1\langle F^R_{\mu\nu} F^{R\mu\nu} + F^L_{\mu\nu} F^{L\mu\nu}\rangle + H_2\langle \chi^\dagger \chi\rangle
\end{aligned}$$

L_i: Low-energy-constants (LECs)
H_i: Values depend on definition of currents/densities

These absorb the divergences of loop diagrams: $L_i \to L_i^r$
Renormalization: order by order in the powercounting

Lagrangians

Lagrangian Structure:

	2 flavour		3 flavour		3+3 PQChPT	
p^2	F, B	2	F_0, B_0	2	F_0, B_0	2
p^4	l_i^r, h_i^r	7+3	L_i^r, H_i^r	10+2	$\hat{L}_i^r, \hat{H}_i^r$	11+2
p^6	c_i^r	52+4	C_i^r	90+4	K_i^r	112+3

p^2: Weinberg 1966

p^4: Gasser, Leutwyler 84,85

p^6: JB, Colangelo, Ecker 99,00

Chiral Logarithms

The main predictions of ChPT:

- Relates processes with different numbers of pseudoscalars
- Chiral logarithms (and perturbative FSI,. . .)

$$m_\pi^2 = 2B\hat{m} + \left(\frac{2B\hat{m}}{F}\right)^2 \left[\frac{1}{32\pi^2}\log\frac{(2B\hat{m})}{\mu^2} + 2l_3^r(\mu)\right] + \cdots$$

$M^2 = 2B\hat{m}$

$B \neq B_0$, $F \neq F_0$ (two versus three-flavour)

LECs and μ

$l_3^r(\mu)$

$$\bar{l}_i = \frac{32\pi^2}{\gamma_i}\, l_i^r(\mu) - \log\frac{M_\pi^2}{\mu^2}\,.$$

Independent of the scale μ.

For 3 and more flavours, some of the $\gamma_i = 0$: $L_i^r(\mu)$

μ :

- m_π, m_K: chiral logs vanish
- pick larger scale
- 1 GeV then $L_5^r(\mu) \approx 0$ large N_c arguments????
- compromise: $\mu = m_\rho = 0.77$ GeV

Expand in what quantities?

- Expansion is in momenta and masses
- But is not unique: relations between masses (Gell-Mann–Okubo) exists
- Express orders in terms of physical masses and quantities (F_π, F_K)?
- Express orders in terms of lowest order masses?
- E.g. $s + t + u = 2m_\pi^2 + 2m_K^2$ in πK scattering
- Relative sizes of order p^2, p^2, p^4,. . . can vary considerably
- I prefer physical masses
- Thresholds correct
- Chiral logs are from physical particles propagating

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LECs

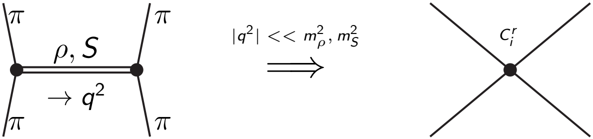
Some combinations of order p^6 LECs are known as well: curvature of the scalar and vector formfactor, two more combinations from $\pi\pi$ scattering (implicit in b_5 and b_6)

General observation:

- Obtainable from kinematical dependences: known
- Only via quark-mass dependence: poorly known

C_i^r

Most analysis use (i.e. almost all of mine):
C_i^r from (single) resonance approximation



Motivated by large N_c: large effort goes in this

Ananthanarayan, JB, Cirigliano, Donoghue, Ecker, Gamiz, Golterman, Kaiser, Knecht, Peris, Pich, Prades, Portoles, de Rafael,...

C_i^r

$$\begin{aligned}\mathcal{L}_V &= -\frac{1}{4}\langle V_{\mu\nu}V^{\mu\nu}\rangle + \frac{1}{2}m_V^2\langle V_\mu V^\mu\rangle - \frac{f_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle\\ &\quad -\frac{ig_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu,u^\nu]\rangle + f_\chi\langle V_\mu[u^\mu,\chi_-]\rangle\\ \mathcal{L}_A &= -\frac{1}{4}\langle A_{\mu\nu}A^{\mu\nu}\rangle + \frac{1}{2}m_A^2\langle A_\mu A^\mu\rangle - \frac{f_A}{2\sqrt{2}}\langle A_{\mu\nu}f_-^{\mu\nu}\rangle\\ \mathcal{L}_S &= \frac{1}{2}\langle\nabla^\mu S\nabla_\mu S - M_S^2S^2\rangle + c_d\langle Su^\mu u_\mu\rangle + c_m\langle S\chi_+\rangle\\ \mathcal{L}_{\eta'} &= \frac{1}{2}\partial_\mu P_1\partial^\mu P_1 - \frac{1}{2}M_{\eta'}^2P_1^2 + i\tilde{d}_m P_1\langle\chi_-\rangle\,.\end{aligned}$$

$f_V = 0.20,\ f_\chi = -0.025,\ g_V = 0.09,\ c_m = 42$ MeV, $c_d = 32$ MeV, $\tilde{d}_m = 20$ MeV, $m_V = m_\rho = 0.77$ GeV, $m_A = m_{a_1} = 1.23$ GeV, $m_S = 0.98$ GeV, $m_{P_1} = 0.958$ GeV

f_V, g_V, f_χ, f_A: experiment
c_m and c_d from resonance saturation at $\mathcal{O}(p^4)$

C_i^r

Problems:

- Weakest point in the numerics
- However not all results presented depend on this
- Unknown so far: C_i^r in the masses/decay constants and how these effects correlate into the rest
- No μ dependence: obviously only estimate

What we do/did about it:

- Vary resonance estimate by factor of two
- Vary the scale μ at which it applies: 600-900 MeV
- Check the estimates for the measured ones
- Again: kinematic can be had, quark-mass dependence difficult

L_i^r and C_i^r

Full NNLO fits of the L_i^r

- Amorós,JB,Talavera, 2000, 2001 (fit 10)
  simple C_i^r
- JB, Jemos, 2011
  simple C_i^r
- JB,Ecker,2014, to be published
  Continuum fit with more input for C_i^r
- Numerics presented for $\eta \to 3\pi$ is with fit 10
  JB,Ghorbani, 2007
- Would expect no major changes from that

Lowest order

ChPT:Cronin 67: $A(s,t,u) = \frac{B_0(m_u - m_d)}{3\sqrt{3}F_\pi^2}\left\{1 + \frac{3(s-s_0)}{m_\eta^2 - m_\pi^2}\right\}$

with $Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$ or $R \equiv \frac{m_s - \hat{m}}{m_d - m_u}$ $\hat{m} = \frac{1}{2}(m_u + m_d)$

$$A(s,t,u) = \frac{1}{Q^2}\frac{m_K^2}{m_\pi^2}(m_\pi^2 - m_K^2)\frac{\mathcal{M}(s,t,u)}{3\sqrt{3}F_\pi^2},$$

$$A(s,t,u) = \frac{\sqrt{3}}{4R}M(s,t,u)$$

LO: $\mathcal{M}(s,t,u) = \frac{3s - 4m_\pi^2}{m_\eta^2 - m_\pi^2}$ $M(s,t,u) = \frac{1}{F_\pi^2}\left(\frac{4}{3}m_\pi^2 - s\right)$

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$\eta \to 3\pi$: p^2 and p^4

- $\Gamma(\eta \to 3\pi) \propto |A|^2 \propto Q^{-4}$ allows a PRECISE measurement
- Q^2 form lowest order mass relation: $\boxed{Q \approx 24}$
  $\Longrightarrow \Gamma(\eta \to \pi^+\pi^-\pi^0)_{\mathrm{LO}} \approx 66$ eV
- $m^2_{K^+} - m^2_{K^0} \sim Q^{-2}$ at NNLO: $\boxed{Q = 20.0 \pm 1.5}$
  $\Longrightarrow \Gamma(\eta \to \pi^+\pi^-\pi^0)_{\mathrm{LO}} \approx 140$ eV
- At order p^4 Gasser-Leutwyler 1985: $\dfrac{\displaystyle\int dLIPS|A_2 + A_4|^2}{\displaystyle\int dLIPS|A_2|^2} = 2.4\,,$

  ($LIPS$=Lorentz invariant phase-space)

  Major source: large S-wave final state rescattering
- Experiment: 300 ± 12 eV (PDG 2012/13)

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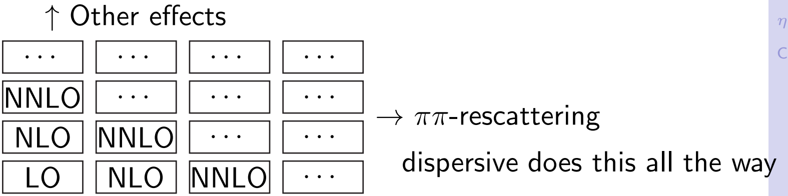
$\eta \to 3\pi$: LO, NLO, NNLO, NNNLO,...

- IN Gasser,Leutwyler, 1985 ($\sqrt{2.4} = 1.55$):
  about half: $\pi\pi$-rescattering
  other half: everything else
- $\pi\pi$-rescattering important Roiesnel, Truong, 1981
- Dispersive approach (next talk): resum all $\pi\pi$
- assume rescattering + rest separable:



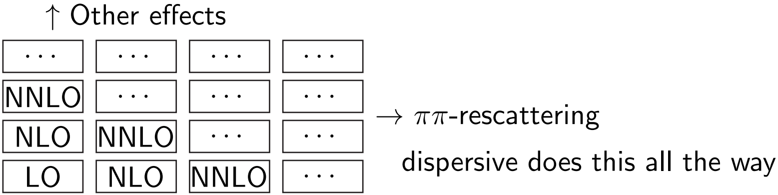
↑ Other effects

$\cdots$	$\cdots$	$\cdots$	$\cdots$
NNLO	$\cdots$	$\cdots$	$\cdots$
NLO	NNLO	$\cdots$	$\cdots$
LO	NLO	NNLO	$\cdots$

→ $\pi\pi$-rescattering

dispersive does this all the way

Why look at it this way?



↑ Other effects

⋯	⋯	⋯	⋯
NNLO	⋯	⋯	⋯
NLO	NNLO	⋯	⋯
LO	NLO	NNLO	⋯

→ $\pi\pi$-rescattering

dispersive does this all the way

- $\delta_\pi = 0.3$, $\delta_O = 0.3$
- LO $= 1$
- NLO $= \delta_\pi + \delta_O = 0.6$
- NNLO $= \delta_\pi^2 + \delta_\pi \delta_O + \delta_O^2 = 0.27$
- Squared: $1 \to 2.6 \to 3.5$
- Underlying other is: $1 + 0.3 + 0.09$
- Goal: remove dispersive from ChPT, then add again via dispersion relations (but now all boxes)
- Problem: Separation is not trivial

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NNLO	$\cdots$	$\cdots$	$\cdots$
NLO	NNLO	$\cdots$	$\cdots$
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Two Loop Calculation: why?

- In $K_{\ell 4}$ dispersive gave about half of p^6 in amplitude
- Same order in ChPT as masses for consistency check on m_u/m_d
- Check size of 3 pion dispersive part
- At order p^4 unitarity about half of correction
- Technology exists:
  - Two-loops: Amorós,JB,Dhonte,Talavera,. . .
  - Dealing with the mixing π^0-η: Amorós,JB,Talavera 01
- Done: JB, Ghorbani, arXiv:0709.0230
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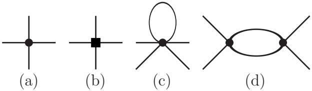
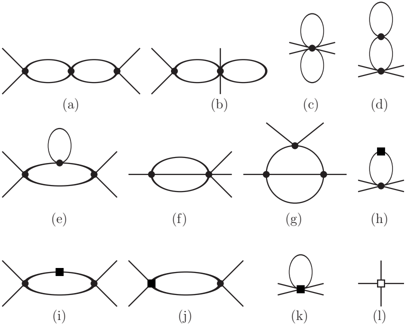
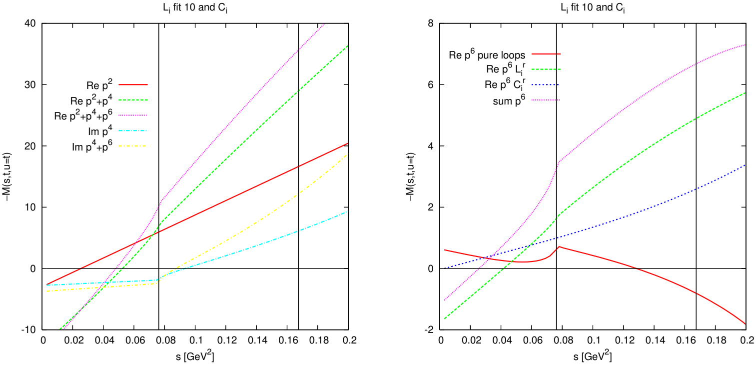
Diagrams





- Include mixing, renormalize, pull out factor $\frac{\sqrt{3}}{4R}$, . . .
- Two independent calculations (comparison lots of work)
- You have to carefully define which LO ($\mathcal{M}$ or M)
- You have to carefully define which NLO
- Integrals only in numerical form: (g) is the hardest one

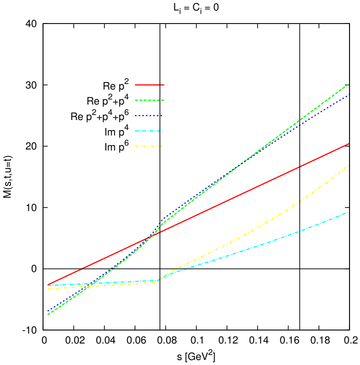
$\eta \to 3\pi$: $M(s, t = u)$

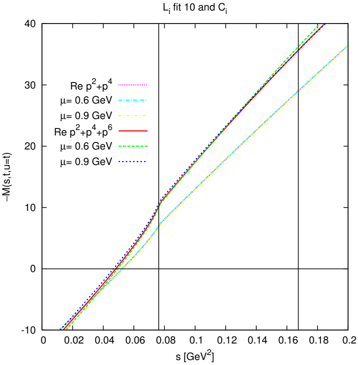


Along $t = u$

Along $t = u$ parts

$\eta \to 3\pi$: $M(s, t = u)$





Along $t = u$

$L_i^r = C_i^r = 0$

Along $t = u$: μ dependence

I.e. where $C_i^r(\mu)$ estimated

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Neutral decay

	$\overline{A}_0^2$	α
LO	1090	0.000
NLO	2810	0.013
NLO ($L_i^r = 0$)	2100	0.016
NNLO	4790	0.013
NNLOq	4790	0.014
NNLO ($C_i^r = 0$)	4140	0.011
NNLO ($L_i^r = C_i^r = 0$)	2220	0.016
dispersive (KWW)	—	$-(0.007$—$0.014)$
tree dispersive	—	-0.0065
absolute dispersive	—	-0.007
Borasoy	—	-0.031
error	160	0.032

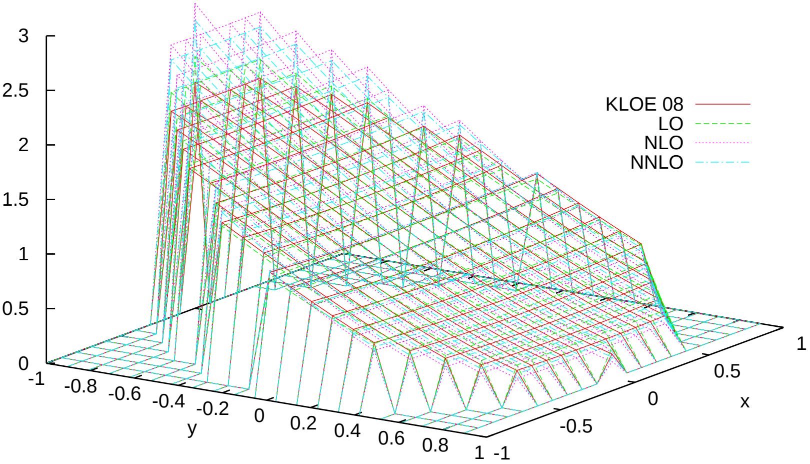
- experiment: $\alpha = -0.032$ with small error
- NNLO ChPT gets a_0^0 in $\pi\pi$ correct

Theory: charged

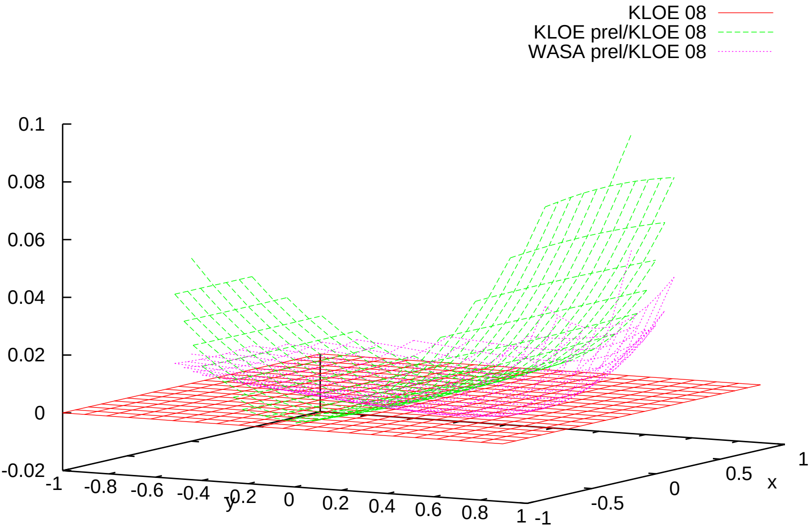
	A_0^2	a	b	d	f
LO	120	-1.039	0.270	0.000	0.000
NLO	314	-1.371	0.452	0.053	0.027
NLO ($L_i^r = 0$)	235	-1.263	0.407	0.050	0.015
NNLO	538	-1.271	0.394	0.055	0.025
NNLOp (y from T^0)	574	-1.229	0.366	0.052	0.023
NNLOq (incl $(x,y)^4$)	535	-1.257	0.397	0.076	0.004
NNLO ($\mu = 0.6$ GeV)	543	-1.300	0.415	0.055	0.024
NNLO ($\mu = 0.9$ GeV)	548	-1.241	0.374	0.054	0.025
NNLO ($C_i^r = 0$)	465	-1.297	0.404	0.058	0.032
NNLO ($L_i^r = C_i^r = 0$)	251	-1.241	0.424	0.050	0.007
dispersive (KWW)	—	-1.33	0.26	0.10	—
tree dispersive	—	-1.10	0.33	0.001	—
absolute dispersive	—	-1.21	0.33	0.04	—
error	18	0.075	0.102	0.057	0.160
KLOE 08		-1.090	0.124	0.057	0.14

- NLO to NNLO changes, but no large ones
- Error: $\Delta|M(s,t,u)|^2 = |M^{(6)}\, M(s,t,u)|$

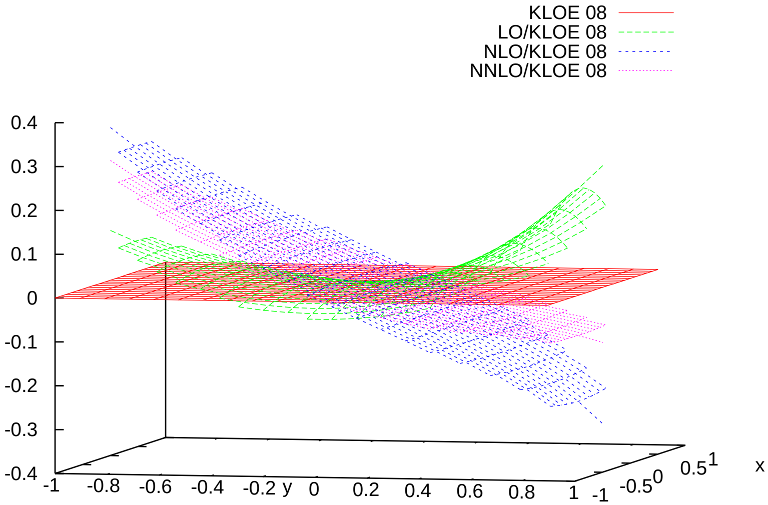
Experiment vs Theory: charged



Experiment : relative



Experiment vs Theory: relative



r and decay rates

$$\sin\epsilon = \frac{\sqrt{3}}{4R} + \mathcal{O}(\epsilon^2)$$

$$\begin{aligned}
\Gamma(\eta\to\pi^+\pi^-\pi^0) &= \sin^2\epsilon \cdot 0.572\ \mathrm{MeV} && \mathrm{LO}\,,\\
&\quad \sin^2\epsilon \cdot 1.59\ \mathrm{MeV} && \mathrm{NLO}\,,\\
&\quad \sin^2\epsilon \cdot 2.68\ \mathrm{MeV} && \mathrm{NNLO}\,,\\
&\quad \sin^2\epsilon \cdot 2.33\ \mathrm{MeV} && \mathrm{NNLO}\ C_i^r = 0\,,\\
\Gamma(\eta\to\pi^0\pi^0\pi^0) &= \sin^2\epsilon \cdot 0.884\ \mathrm{MeV} && \mathrm{LO}\,,\\
&\quad \sin^2\epsilon \cdot 2.31\ \mathrm{MeV} && \mathrm{NLO}\,,\\
&\quad \sin^2\epsilon \cdot 3.94\ \mathrm{MeV} && \mathrm{NNLO}\,,\\
&\quad \sin^2\epsilon \cdot 3.40\ \mathrm{MeV} && \mathrm{NNLO}\ C_i^r = 0\,.
\end{aligned}$$

r and decay rates

$$r \equiv \frac{\Gamma(\eta \to \pi^0\pi^0\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)}$$

$$\begin{aligned} r_{\mathrm{LO}} &= 1.54 \\ r_{\mathrm{NLO}} &= 1.46 \\ r_{\mathrm{NNLO}} &= 1.47 \\ r_{\mathrm{NNLO}\, C_i^r=0} &= 1.46 \end{aligned}$$

PDG 2013

$$\begin{aligned} r &= 1.48 \pm 0.05 \quad \text{our average}\,. \\ r &= 1.426 \pm 0.026 \quad \text{our fit}\,, \end{aligned}$$

Reasonable agreement

R and Q from $eta \to 3\pi$

	LO	NLO	NNLO	NNLO ($C_i^r = 0$)
R (η)	18.9	31.5	40.9	38.2
R (Dashen)	44	44	37	—
R (Dashen-violation)	36	37	32	—
Q (η)	16.5	21.3	24.3	23.4
Q (Dashen)	24	24	22	—
Q (Dashen-violation)	22	22	20	—

LO from $R = \dfrac{m_{K^0}^2 + m_{K^+}^2 - 2m_{\pi^0}^2}{2\left(m_{K^0}^2 - m_{K^+}^2\right)}$ (QCD part only)

NLO and NNLO from masses: Amorós, JB, Talavera 2001

$$Q^2 = \frac{m_s + \hat{m}}{2\hat{m}} R = 14.4R$$

($m_s/\hat{m} = 27.8$ used for $\eta \to 3\pi$)

$\eta \to \pi^0\gamma\gamma$

- Not a full review, pointing out certain things
- Long ago: $\gamma\gamma \to \pi^0\pi^0$ only loop at order p^4

  JB, Cornet, 1988, Donoghue, Holstein, Lin, 1988
- Same argument goes for $\eta \to \pi^0\gamma\gamma$
  - Neutral particles and gauge invariance: $A \propto F_{\mu\nu}F^{\mu\nu}$
  - But chiral invariance requires U and $U^\dagger$ and neutral parts commute with Q
- Ametller, JB, Bramon, Cornet, Phys. Lett. B **276** (1992) 185
- π-loop suppressed by $m_u - m_d$
- K-loop very small: the loop integral $s_{\gamma\gamma}$ small is $\frac{1}{24m_K^2}$
- total at p^4:

  π-loops: 0.00084 eV
  K-loops: 0.00245 eV
  sum: 0.00389 eV
- Note that this decay does not break isospin

Experiment

Large background from $\eta \to 3\pi^0$ with missed photons

	$10^4 BR$	$\Gamma(\eta \to \pi^0\gamma\gamma)$ [eV]
GAMS 2000	7.1 ± 1.4	0.92 ± 0.18
CBALL (BNL)	2.2 ± 0.5	0.29 ± 0.07
CBALL (MAMI-B)	$2.21 \pm 0.24 \pm 0.47$	0.29 ± 0.07
KLOE	$0.84 \pm 0.27 \pm 0.14$	0.11 ± 0.04

Results on the spectrum exists: talk by Unverzagt

General structure

- The amplitude has two structures: A and B $\eta(P) \to \pi^0(p)\gamma(q_1)\gamma_(q_2)$:

$$\begin{aligned} M &= A[(\epsilon_1\cdot\epsilon_2)(q_1\cdot q_2) - (q_2\cdot\epsilon_1)(q_1\cdot\epsilon_2)] \\ &+ B[-(\epsilon_1\cdot\epsilon_2)(P\cdot q_1)(P\cdot q_2) - (P\cdot\epsilon_1)(P\cdot\epsilon_2)(q_1\cdot q_2) \\ &\qquad +(P\cdot\epsilon_1)(q_1\cdot\epsilon_2)(P\cdot q_1) + (q_2\cdot\epsilon_1)(P\cdot\epsilon_2)(P\cdot q_2)] \end{aligned}$$

- At p^4 only A nonzero
- At order p^6 pure pion diagrams suppressed by $m_u - m_d$
- There exists a partial two-loop calculation (i.e. difficult 2-loop part not done) Jetter hep-ph/9508407 Finds the expected small corrections from the loops
- The decay rate is quite sensitive to interferences

Moreloops

- First pure pion loop contribution that has no isospin breaking is with a double WZW term (i.e. p^8)
- WZW^2:

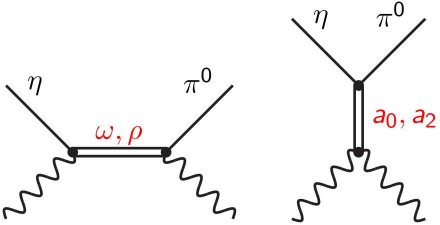
π-loops:	0.00005 eV
K-loops:	0.0022 eV
sum:	0.0025 eV

- Same size as p^4 loops
- Better look for other contributions
- But also a way to check a nontrivial process with resonance saturation

What else should we look at?

- Higher order ChPT LECs dominated by resonances
  Gasser,Leutwyler 84, Ecker, Gasser, Pich, de Rafael 1989,

  Donoghue, Ramirez, Valencia 1989
- Possible major contributions (isospin conserving):



- Determine the couplings from experiment/other theory (I will use 1992 numbers)
- Keep full propagators and/or restrict to p^6 part only
- Models might have more contributions

Fix couplings

- $VP\gamma$ vertex from $\omega \to \pi\gamma$
- The product of $a_0\pi^0\eta$ and $a_0\gamma\gamma$ from $\gamma\gamma \to \pi^0\eta$
- Same with $a_0 \to a_2$ (and choose an off-shell a_2 propagator)
- Note that much of these data have changed since 1992 (but reasonably compatible)

Results

Included	$\Gamma(\eta \to \pi^0\gamma\gamma)$ [eV]	
ρ^0, ω p^6	0.18	
ρ^0, ω	0.31	VMD
ρ^0, ω, a_0, a_2 p^6	0.16-0.20	interference sign not known
ρ^0, ω, π, K	0.42	parts with fixed signs
$+a_0, a_2$	0.35-0.50	signs again
+uncertainty	0.22-0.62	large N_c, p^6 and other loops

We concluded $\Gamma(\eta \to \pi^0\gamma\gamma) = 0.42 \pm 0.20$ eV

Other models

- Chiral quark model to p^6: about 0.15 eV
  JB,Dawson,Valencia,1991 (same counterterm as $\gamma\gamma \to \pi^0\pi^0$)
- ENJL and variations is very popular
  some are: Bel'kov,Lanyov, Scherer, 1995, Bellucci, Bruno 1995, JB, Fayyazzudin, Prades, 1996, Nemoto, Oka, Takizawa 96, Radzhabov, Volkov 2006

  Numbers last reference; models differ in η-η'-mixing

	model 1	model 2
vector mesons	0.17	0.20
scalar meson	0.03	0.12
vector+scalar	0.10	0.12
box	0.28	0.35
box+vector	0.78	0.95
total	0.53	0.45

- Interferences important

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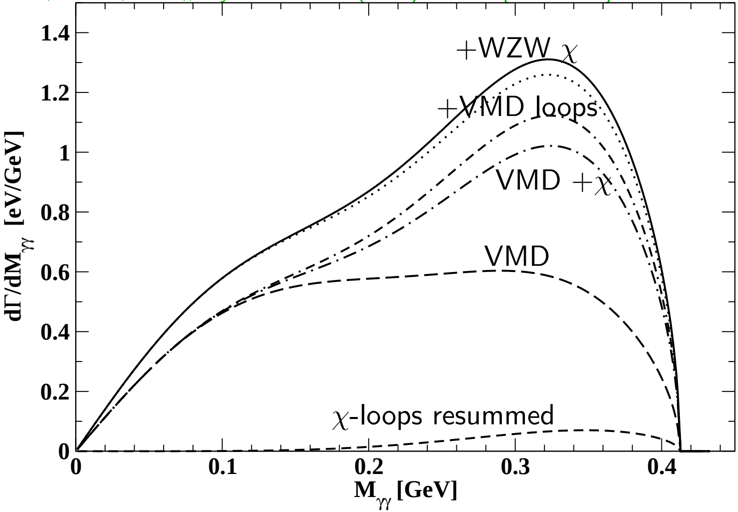
Other approaches/models

- Use the chiral unitary approach Oset, Pelaez, Roca, 2003,2008
- Generates the a_0 dynamically and thus fixes its signs $(+)$
- Also included newer $V \to P\gamma$ experiments
- $\Gamma(\eta \to \pi^0\gamma\gamma) = 0.33 \pm 0.08$ eV
- I think a_2 is still missing
- There are also purely dispersive approaches

Distributions

Oset, Pelaez, Roca,,Phys. Rev. D **77** (2008) 073001 [0801.2633]



Conclusions: Model testing good with spectra

$M_{\pi^0\gamma}$ less sensitive but can help

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Conclusions

- Overview of ChPT for $\eta \to 3\pi$
  - ChPT at two loops has sizable but not unusual corrections
  - Does not reproduce at present the Dalitz plot
  - If married with dispersive, indications that the 'other' series might work well
- Overview of $\eta \to \pi^0\gamma\gamma$
  - Chiral loops are small (if not resummed to generate resonances)
  - VMD is main part
  - Others do help: note scalars are important and their signs