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# Dispersive Approach To The Gamma－W Box Diagram In Beta Decay 

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## Outline

## 1.Motivation <br> 2. Radiative Corrections to Beta Decay <br> 3. W-Gamma Box Diagram <br> 4. Dispersion Relation <br> 5.Brief Summary

In collaboration with M. J. Ramsey-Musolf, M. Gorshteyn, H. Patel, C. P. Liu and C. W. Gao.

## Motivation

- Beta decay has long been a perfect venue for precision test of SM as well as search of BSM physics.
- In order to extract definite conclusions from experiments, we need to know the error budget in SM prediction. (e.g. talks by Ramsey-Musolf and Hardy)
- Super-allowed beta decays are powerful in the determination of the CKM matrix element $\mathrm{V}_{\mathrm{ud}}$; neutron beta decay is less competitive for $\mathrm{V}_{\mathrm{ud}}$ but still powerful in probing new physics due to the existence of extra structures:
$\frac{d \Gamma}{d E_{e} d \Omega_{e} d \Omega_{v}}=\frac{\left(G_{F} V_{u d}\right)^{2}}{(2 \pi)^{5}}\left(g_{V}^{2}+3 g_{A}^{2}\right)\left|\vec{p}_{e}\right| E_{e} E_{v}^{2}\left(1+a \frac{\vec{p}_{e} \cdot \vec{p}_{v}}{E_{e} E_{v}}+b \frac{m_{e}}{E_{e}}+\hat{s}_{n} \cdot\left(A \frac{\vec{p}_{e}}{E_{e}}+B \frac{\vec{p}_{v}}{E_{v}}+D \frac{\vec{p}_{e} \times \vec{p}_{v}}{E_{e} E_{v}}\right)\right)$
- Future experiments in neutron beta decay are aiming for precision level of $10^{-3}-10^{-4}$ (see experimental talks), i.e. all higher-order corrections up to order $\alpha / 4 \pi$ (on top of the LO contribution) should be precisely determined before one can really make use of such results.


## Radiative Corrections to Beta Decay

- Classification of one-loop structures that contribute to beta decay:

1) Three-current correlation diagrams
2) Diagrams with three-boson vertex
3) W- $\gamma$ box diagram
4) W-Z box diagrams
5) Diagrams with Higgs bosons
6) Lepton-boson vertex corrections and self-energy diagrams

- Current Algebra and Operator Product Expansion (OPE) are useful tools to isolate $O\left(G_{F} \alpha\right)$ contributions

Sirlin, Rev.Mod.Phys 50 (1978) 573 (and references therein)
Marciano and Sirlin, Phys.Rev.Lett. 56 (1985) 22
Marciano and Sirlin, Phys.Rev.Lett. 96 (2006) 032002

- Effective Field Theory (EFT) as an alternative approach

Ando et al, Phys.Lett.B. 595 (2004) 250

## Radiative Corrections to Beta Decay

- $\quad$ SOP in the current algebra analysis:

1. Use equal-time commutation relations to reduce the number of currents in correlation functions.

$$
\begin{aligned}
& {\left[J_{W}^{0}(x), J_{Z}^{\mu}\left(x^{\prime}\right)\right]_{x^{0}=x^{0}}=\cos ^{2} \theta_{W} J_{W}^{\mu}(x) \delta^{3}\left(\vec{x}-\vec{x}^{\prime}\right)} \\
& {\left[J_{W}^{0}(x), J_{\gamma}^{\mu}\left(x^{\prime}\right)\right]_{x^{0}=x^{0}}=J_{W}^{\mu}(x) \delta^{3}\left(\vec{x}-\vec{x}^{\prime}\right)} \\
& {\left[J_{W}^{0}(x), J_{W}^{+\mu}\left(x^{\prime}\right)\right]_{x^{0}=x^{00}}=-2\left(\sin ^{2} \theta_{W} J_{\gamma}^{\mu}(x)+J_{Z}^{\mu}(x)\right) \delta^{3}\left(\vec{x}-\vec{x}^{\prime}\right)+S . T}
\end{aligned}
$$

2. Count the number $n$ of massive propagators. If $n \geq 2$ then the contribution from IR region of the loop momentum is negligible.
3. For two-current correlator contributions in UV region, perform OPE to the current product. Effects of higher-order terms are suppressed by inverse powers of $\mathrm{m}_{\mathrm{W}}$.
4. One-current matrix element of the charged weak current is proportional to the zeroth order amplitude.

## Radiative Corrections to Beta Decay

- One example to demonstrate the steps. Consider the following diagram:


$$
T_{(b)}^{\lambda \rho}(k)=\int d^{4} x e^{i k \cdot x}\left\langle h^{\prime}\right| T\left[J_{b}^{\lambda}(x) J_{W}^{\rho}(0)\right]|h\rangle
$$

$$
i M \sim \frac{i}{q^{2}-m_{W}^{2}} \bar{u}_{e L} \gamma^{\mu} V_{e L} \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k^{2}-m_{b}^{2}} \frac{1}{(k-q)^{2}-m_{W}^{2}} V_{\mu \lambda \rho}(k, q) T_{(b)}^{\lambda \rho}(k)
$$

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$$

- Since there are at least two heavy propagators, for small values of k the integral is suppressed by $\left(1 / \mathrm{m}_{\mathrm{W}}\right)^{4}$ or more and can be neglected.


## Radiative Corrections to Beta Decay

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T_{(b)}^{\lambda \rho}(k)=\int d^{4} x e^{i k \cdot x}\left\langle h^{\prime}\right| T\left[J_{b}^{\lambda}(x) J_{W}^{\rho}(0)\right]|h\rangle
$$

$$
i M \sim \frac{i}{q^{2}-m_{W}^{2}} \bar{u}_{e L} \gamma^{\mu} v_{e L} \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k^{2}-m_{b}^{2}} \frac{1}{(k-q)^{2}-m_{W}^{2}} V_{\mu \lambda \rho}(k, q) T_{(b)}^{\lambda \rho}(k)
$$

- For large values of k one could perform OPE to the hadronic tensor. Examples of free-field OPE:

$$
\begin{aligned}
& T_{(\gamma) \lambda}^{\lambda}(k)=\frac{2 i k^{\mu}}{k^{2}}\left\langle h^{\prime}\right| J_{W \mu}(0)|h\rangle+\ldots \\
& T_{(Z) \lambda}^{\lambda}(k)=\frac{2 i k^{\mu}}{k^{2}} \cos ^{2} \theta_{W}\left\langle h^{\prime}\right| J_{W \mu}(0)|h\rangle+\ldots
\end{aligned}
$$

## Radiative Corrections to Beta Decay

- One example to demonstrate the steps. Consider the following diagram:


$$
T_{(b)}^{\lambda \rho}(k)=\int d^{4} x e^{i k \cdot x}\left\langle h^{\prime}\right| T\left[J_{b}^{\lambda}(x) J_{W}^{\rho}(0)\right]|h\rangle
$$

$$
i M \sim \frac{i}{q^{2}-m_{W}^{2}} \bar{u}_{e L} \gamma^{\mu} v_{e L} \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k^{2}-m_{b}^{2}} \frac{1}{(k-q)^{2}-m_{W}^{2}} V_{\mu \lambda \rho}(k, q) T_{(b)}^{\lambda \rho}(k)
$$

- The one-current matrix element is a constant and the remaining integral over k can be carried out analytically.


## Radiative Corrections to Beta Decay

- One example to demonstrate the steps. Consider the following diagram:


$$
T_{(b)}^{\lambda \rho}(k)=\int d^{4} x e^{i k \cdot x}\left\langle h^{\prime}\right| T\left[J_{b}^{\lambda}(x) J_{W}^{\rho}(0)\right]|h\rangle
$$

$$
i M \sim \frac{i}{q^{2}-m_{W}^{2}} \bar{u}_{e L} \gamma^{\mu} v_{e L} \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k^{2}-m_{b}^{2}} \frac{1}{(k-q)^{2}-m_{W}^{2}} V_{\mu \lambda \rho}(k, q) T_{(b)}^{\lambda \rho}(k)
$$

- Combining with the three-current correlator contributions (see next page), their effects are either finite or can be absorbed into the renormalization of weak coupling constant. Both of them are proportional to the zeroth order amplitude.


## Radiative Corrections to Beta Decay

- Three-current correlation diagrams:


$$
\begin{aligned}
& J_{W}^{\mu}=\bar{q}_{L} \gamma^{\mu} \tau^{+} q_{L} \\
& J_{\gamma}^{\mu}=\bar{q} \gamma^{\mu} Q q \\
& J_{Z}^{\mu}=\frac{1}{2} \bar{q}_{L} \gamma^{\mu} \tau_{3} q_{L}-\sin ^{2} \theta_{W} J_{\gamma}^{\mu}
\end{aligned}
$$

$$
T_{(b)}^{\mu}(\bar{q}, k) \sim \int d^{4} y e^{i \bar{q} \cdot y} \int d^{4} x e^{i k \cdot x}\left\langle h^{\prime}\right| T\left[J_{W}^{\mu}(y)\left(J_{b}^{+\lambda}(x) J_{b \lambda}(0)+h . c .\right)\right]|h\rangle
$$

Current Algebra Reduction
$\int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k^{2}-m_{b}^{2}} \frac{\partial}{\partial k_{\mu}} T_{(b) \lambda}^{\lambda}(k)$

$$
\int d^{4} y e^{i \bar{q} \cdot y} \int d^{4} x e^{i k \cdot x}\left\langle h^{\prime}\right| T\left[\partial \cdot J_{W}(y)\left(J_{b}^{+\lambda}(x) J_{b \lambda}(0)+h . c .\right)\right]|h\rangle
$$

Residual three-current correlator:
Contributions are of order $\mathrm{O}\left(\mathrm{G}_{\mathrm{F}}{ }^{2}\right)$

## Radiative Corrections to Beta Decay

- W-Z box diagrams:

- Two massive propagators $\rightarrow$ only UV region is important. OPE can be performed and only the leading term is retained.
- The contribution is proportional to the zeroth order amplitude


## Radiative Corrections to Beta Decay

- Other less important RC diagrams:
- Diagrams with Higgs: effects are $\mathrm{O}\left(\mathrm{G}_{\mathrm{F}}{ }^{2}\right)$.
- Lepton-gauge vertex corrections and self-energy diagrams: their effects are universal (i.e. same in hadron/lepton beta decay) and also well-studied.
- So far, all the $O\left(G_{F} \alpha\right)$ effects we studied can always be reduced to the form of the zeroth order amplitude, either by current algebra or by leading-order OPE. The former is interaction-independent, while the latter is modified by perturbative QCD, which effects can be systematically included.


## W-Gamma Box Diagram

- $W-\gamma$ box diagram:

- Current algebra + combining with the two-current term derived from three-current correlator:

$$
\begin{aligned}
& i M_{(\gamma)}=\frac{-i g^{2} e^{2} V_{u d}}{2(2 \pi)^{4} m_{W}^{2}} \bar{u}_{e L} \gamma^{\mu} v_{v L} \int \frac{d^{4} k}{k^{4}} \frac{m_{W}^{2}}{m_{W}^{2}-k^{2}}\left\{-2 i\left\langle h^{\prime}\right| J_{W \mu}(0)|h\rangle\right. \\
& \left.+i \varepsilon_{\lambda \rho \sigma \mu} k^{\sigma} T_{(\gamma)}^{\lambda \rho}(k)-i \int d^{4} x e^{i k \cdot x}\left\langle h^{\prime}\right| T\left[J_{\gamma \mu}(x) \partial \cdot J_{W}(0)\right]|p\rangle\right\}
\end{aligned}
$$

- Only one massive propagator. Both IR and UV region could give $\mathrm{O}\left(\mathrm{G}_{\mathrm{F}} \alpha\right)$ corrections.
- Should combine with real emission diagrams to cancel IR divergences.


## W-Gamma Box Diagram

- $\quad \mathrm{RC}$ to the Fermi amplitude: most uncertainty comes from the $\mathrm{V}^{*} \mathrm{~A}$ current correlator:

$$
\begin{aligned}
& i M_{(\gamma ; A)}^{\mathrm{box}, F}=\frac{-g^{2} e^{2} V_{u d}}{4(2 \pi)^{4}} \varepsilon_{\lambda \rho \sigma \mu} \bar{u}_{e L} \gamma^{\mu} V_{v J} \int \frac{d^{4} k}{k^{4}} \frac{k^{\sigma}}{m_{W}^{2}-k^{2}} A^{\lambda \rho}(k), \\
& A^{\lambda \rho}(k)=\int d^{4} x e^{i k \cdot x}\left\langle h^{\prime}\right| T\left[J_{\gamma}^{\lambda}(x) A_{W}^{\rho}(0)\right]|h\rangle
\end{aligned}
$$

- The asymptotic correction can be obtained by OPE:

$$
\begin{gathered}
A^{\lambda \rho}(k)=-\frac{1}{3} \varepsilon^{\lambda \rho \alpha \beta} \frac{k_{\alpha}}{k^{2}}\left\langle h^{\prime}\right| V_{W \beta}(0)|h\rangle+\ldots \\
i M_{(\gamma ; A)}^{\mathrm{box}, F}=i M^{0, F} \frac{\alpha}{4 \pi}\left[\ln \left(\frac{m_{Z}}{m_{A}}\right)+A_{g}+2 C\right]
\end{gathered}
$$

- The evaluation of the non-asymptotic contribution is a real challenge.


## W-Gamma Box Diagram

- State-of-the-art study of box contribution:

$$
i M_{(\gamma ; A)}^{\mathrm{box}, F}=i M^{0, F} \frac{\alpha}{8 \pi} \int_{0}^{\infty} d \kappa^{2} \frac{m_{W}^{2}}{\kappa^{2}+m_{W}^{2}} F\left(\kappa^{2}\right)
$$

Marciano and Sirlin, Phys.Rev.Lett. 96 (2006) 032002

1. Short distance: leading OPE + perturbative QCD

$$
F\left(\kappa^{2}\right)=\frac{1}{\kappa^{2}}\left[1-\frac{\alpha_{S}\left(\kappa^{2}\right)_{\overline{M S}}}{\pi}-C_{2}\left(\frac{\alpha_{S}\left(\kappa^{2}\right)_{\overline{M S}}}{\pi}\right)^{2}-C_{3}\left(\frac{\alpha_{S}\left(\kappa^{2}\right)_{\overline{M S}}}{\pi}\right)^{3}\right]
$$

2. Intermediate distance: interpolating function

$$
F\left(\kappa^{2}\right)=\frac{-1.490}{\kappa^{2}+m_{\rho}^{2}}+\frac{6.855}{\kappa^{2}+m_{A}^{2}}-\frac{4.414}{\kappa^{2}+m_{\rho^{\prime}}^{2}} \begin{gathered}
(0.823 \mathrm{GeV})^{2}<\kappa^{2}<(1.5 \mathrm{GeV})^{2} \\
\text { Assigned error: } 100 \%
\end{gathered}
$$

3. Long distance: Born contribution with nucleon EM and axial current dipole FFs:

$$
C_{\text {Borm }}(\text { neutron }) \approx 0.829 \quad 0<\kappa^{2}<(0.823 \mathrm{GeV})^{2}
$$

## Dispersion Relation Approach

- The aim is to extract the $\mathrm{O}\left(\mathrm{G}_{\mathrm{F}} \alpha\right)$ corrections in the box diagram. It is therefore appropriate to set:

$$
l_{e}=l_{v}=m_{e}=m_{n}-m_{p}=0
$$

- Crossing symmetry:
$n\left(p_{n}\right) \rightarrow p\left(p_{p}\right) e\left(l_{e}\right) \bar{\nu}\left(l_{v}\right) \Leftrightarrow n\left(p_{n}\right) v\left(-l_{v}\right) \rightarrow p\left(p_{p}\right) e\left(l_{e}\right)$
Hence it is appropriate to study the "forward scattering" process:

$$
n\left(p_{N}\right) v(l) \rightarrow p\left(p_{N}\right) e(l)
$$

and take $l \rightarrow 0$ at the end.

- Mandelstam variables: $s=\left(p_{N}+l\right)^{2}, t=0, u=\left(p_{N}-l\right)^{2}=2 m_{N}^{2}-s$

The only independent variable can be chosen as:

$$
v=\frac{p_{N} \cdot l}{m_{N}}=\frac{s-m_{N}^{2}}{2 m_{N}}
$$

- Beta decay corresponds to $v \approx 0$.


## Dispersion Relation Approach

- $\quad \mathrm{S}$ - and u -channel singularities:


Branch cut: $\quad s \geq m_{N}^{2} \Rightarrow v \geq 0$


$$
u \geq m_{N}^{2} \Rightarrow s \leq m_{N}^{2} \Rightarrow v \leq 0
$$

- (Unsubtracted) dispersion relation:

$$
M^{\text {box }}(v+i \varepsilon)=\frac{1}{\pi} \int_{-\infty}^{\infty} d v^{\prime} \frac{\operatorname{Im} M^{\text {box }}\left(v^{\prime}+i \varepsilon\right)}{v^{\prime}-v-i \varepsilon}
$$

where

$$
\operatorname{Im} M^{\text {box }}\left(v^{\prime}+i \varepsilon\right)=\frac{1}{2 i}\left(M^{\text {box }}\left(v^{\prime}+i \varepsilon\right)-M^{\text {box }}\left(v^{\prime}-i \varepsilon\right)\right)
$$



## Dispersion Relation Approach

- Discontinuity is obtained by placing the intermediate states on-shell:

$$
\begin{gathered}
\frac{1}{k^{2}} \rightarrow-2 \pi i \delta\left(k^{2}\right) \Theta\left(k_{0}\right), \quad \begin{array}{l}
\text { S-channel: } \quad T_{(\gamma)}^{\mu \nu}(q) \rightarrow W^{\mu \nu}(q) \\
\text { U-channel: } T_{(\gamma)}^{\mu \nu}(-q) \rightarrow-W^{\mu \nu}(-q) \\
\text { where: } \quad W^{\mu \nu}(q)=\int d^{4} x e^{i q \cdot x}\left\langlep _ { N } \left[\left[J_{\gamma}^{\mu}(x), J_{W}^{v}(0)\right]\left|p_{N}\right\rangle\right.\right.
\end{array}, ~
\end{gathered}
$$

- The dispersion representation reads:

$$
\begin{aligned}
& \operatorname{Re} M^{\text {box }}(v)=-\frac{g^{2} e^{2} V_{u d}}{128 \pi^{3} m_{N}} \int_{0}^{\infty} d v^{\prime} \operatorname{Pr} \frac{1}{v^{\prime}\left(v^{\prime 2}-v^{2}\right)} \int_{m_{N}^{2}}^{s^{\prime}} d W^{2} \int_{0}^{Q_{\max }^{2}} d Q^{2} L^{\mu \nu}\left(l^{\prime}, q\right) \times \\
& \frac{1}{Q^{2}} \frac{1}{Q^{2}+m_{W}^{2}}\left[v^{\prime}\left(W_{\mu \nu}(q)-W_{\mu \nu}(-q)\right)+v\left(W_{\mu \nu}(q)+W_{\mu \nu}(-q)\right)\right]
\end{aligned}
$$

$$
\text { Standard notation: } W=p_{N}+q, \quad Q_{\max }^{2}=\frac{\left(s^{\prime}-m_{N}^{2}\right)\left(s^{\prime}-W^{2}\right)}{s^{\prime}}
$$

Lepton tensor: $\quad L^{\mu \nu}\left(l^{\prime}, q\right)=2\left(2 l^{\prime \mu} l^{\nu}-l^{\prime \mu} q^{\nu}-l^{\nu} q^{\mu}-\frac{Q^{2}}{2} g^{\mu \nu}+i \varepsilon^{\mu \nu \alpha \beta} q_{\alpha} l^{\prime}{ }_{\beta}\right)$

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\text { where: } \quad W^{\mu \nu}(q)=\int d^{4} x e^{i q \cdot x}\left\langle p_{N}\right|\left[J_{\gamma}^{\mu}(x), J_{W}^{v}(0)\right]\left|p_{N}\right\rangle
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& \frac{1}{Q^{2}} \frac{1}{Q^{2}+m_{W}^{2}}\left[v^{\prime}\left(W_{\mu \nu}(q)-W_{\mu \nu}(-q)\right)+v\left(W_{\mu \nu}(q)+W_{\mu \nu}(-q)\right)\right]
\end{aligned}
$$

- Caveat: this approach could miss constant subtraction terms!
- Can be used to study the energy-dependence of the decay, but its importance is questioned as it is usually $\mathrm{E} / \mathrm{m}_{\mathrm{h}}$-suppressed.
- Exceptions may occur if there is a new small energy scale in the system.


## Dispersion Relation Approach

- Alternative approach: hybrid between loop integral and dispersion relation of forward scattering amplitude.

$$
i M=-\frac{e^{2} G_{F} V_{u d}}{\sqrt{2}} \int \frac{d^{4} q}{(2 \pi)^{4}} \frac{\bar{u}_{e L} \gamma^{\mu} q \cdot \gamma \gamma^{\nu} v_{V L}}{q^{4}} \frac{m_{W}^{2}}{q^{2}-m_{W}^{2}} T_{(\gamma) \mu v}(q)
$$

- Write the forward Compton tensor in terms of "structure functions":

$$
T_{(\gamma)}^{\mu \nu}\left(p_{N}, q\right)=\left(-g_{\mu \nu}+\frac{q^{\mu} q^{\nu}}{q^{2}}\right) T_{1}+\frac{\hat{p}_{N}^{\mu} \hat{N}_{N}^{v}}{p_{N} \cdot q} T_{2}+\frac{i \varepsilon^{\mu \nu \alpha \beta} p_{N a} q_{\beta}}{2 p_{N} \cdot q} T_{3}+\ldots
$$

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$$

- For each structure function one can write down the dispersion relation:

$$
T_{i}\left(v+i \varepsilon, Q^{2}\right)=\frac{1}{\pi} \int_{-\infty}^{\infty} d v^{\prime} \frac{\operatorname{Im} T_{i}\left(v^{\prime}+i \varepsilon, Q^{2}\right)}{v^{\prime}-v-i \varepsilon} \quad v=p_{N} \cdot q
$$

- The imaginary part can be obtained by optical theorem:

$$
\operatorname{Im} T_{(\gamma)}^{\mu \nu} \sim W^{\mu \nu}
$$



- It does not miss any constant term as we do know $\mathrm{T}_{\mathrm{i}}$ at low energy.


## Dispersion Relation Approach

- Born contribution can be studied as illustration.
- EM and charged weak form factors:

$$
\begin{gathered}
\left\langle N, p^{\prime}\right| J_{\gamma}^{\mu}(0)|N, p\rangle=\bar{u}\left(p^{\prime}\right)\left[F_{1 N}\left(Q^{2}\right) \gamma^{\mu}+i \frac{F_{2 N}\left(Q^{2}\right)}{2 m_{N}} \sigma^{\mu \alpha}\left(p-p^{\prime}\right)_{\alpha}\right] u(p) \\
\left\langle N^{\prime}, p^{\prime}\right| J_{W}^{\mu}(0)|N, p\rangle=\bar{u}\left(p^{\prime}\right)\left[F_{1}^{W}\left(Q^{2}\right) \gamma^{\mu}+i \frac{F_{2}^{W}\left(Q^{2}\right)}{2 m_{N}} \sigma^{\mu \alpha}\left(p-p^{\prime}\right)_{\alpha}+G_{A}\left(Q^{2}\right) \gamma^{\mu} \gamma^{5}+\ldots\right] u(p)
\end{gathered}
$$

- Resulting hadronic tensor:

$$
\begin{aligned}
& W^{\mu \nu}(q)-W^{\mu \nu}(-q)=2 \pi \delta\left(W^{2}-m_{N}^{2}\right) \Theta\left(W_{0}\right) \times \\
& \bar{u}\left(p_{N}\right)\left[\left(F_{1 p}\left(Q^{2}\right) \gamma^{\mu}-i \frac{F_{2 p}\left(Q^{2}\right)}{2 m_{N}} \sigma^{\mu \alpha} q_{\alpha}\right)\left(\gamma \cdot W+m_{N}\right)\left(F_{1}^{W}\left(Q^{2}\right) \gamma^{\nu}+i \frac{F_{2}^{W}\left(Q^{2}\right)}{2 m_{N}} \sigma^{\nu \beta} q_{\beta}+G_{A}\left(Q^{2}\right) \gamma^{\nu} \gamma^{5}\right)\right. \\
& \left.+\left(F_{1}^{W}\left(Q^{2}\right) \gamma^{\nu}-i \frac{F_{2}^{W}\left(Q^{2}\right)}{2 m_{N}} \sigma^{\nu \beta} q_{\beta}+G_{A}\left(Q^{2}\right) \gamma^{\nu} \gamma^{5}\right)\left(\gamma \cdot W+m_{N}\right)\left(F_{1 n}\left(Q^{2}\right) \gamma^{\mu}+i \frac{F_{2 n}\left(Q^{2}\right)}{2 m_{N}} \sigma^{\mu \alpha} q_{\alpha}\right)\right] u\left(p_{N}\right)
\end{aligned}
$$

## Dispersion Relation Approach

- Pion production:

- For electromagnetic pion production:

$$
J_{\gamma}^{\mu}=\frac{1}{3 \sqrt{2}} J_{I=0}^{\mu}+\frac{1}{\sqrt{2}} J_{I=1}^{\mu}+\ldots
$$

- $\quad \mathrm{I}=0$ and $\mathrm{I}=1$ piece can be separated by choosing different isospin configurations for initial/final states.
- Weak pion production could be probed in neutrino scattering. But poor data.
- The vector part can be inferred from I=1 EM pion production. Axial part may need modeling. Should make sure to have advantage over M - S modeling.
- For low energy pions, all these FFs can be studied using ChPT.


## Dispersion Relation Approach

- DIS regime:
- The hadronic tensor: $W^{\mu \nu}(q)=\int d^{4} x e^{i q \cdot x}\left\langle p, p_{N} \mid\left[J_{\gamma}^{\mu}(x), J_{W}^{\nu}(0)\right] n, p_{N}\right\rangle$ can be related to flavor-diagonal tensors through isospin rotation:

$$
\begin{aligned}
& \langle p| J_{\gamma}^{\mu}(x) J_{W}^{v}(0)|n\rangle=2 \sin ^{2} \theta_{W}\langle p| J_{\gamma}^{\mu}(x) J_{\gamma}^{v}(0)|p\rangle+2\langle p| J_{\gamma}^{\mu}(x) J_{z}^{v}(0)|p\rangle-2\langle p|\left(J_{W}^{+\mu}(x)\right)_{V} J_{w}^{v}(0)|p\rangle \\
& \langle p| J_{W}^{v}(x) J_{\gamma}^{\mu}(0)|n\rangle=-2 \sin ^{2} \theta_{W}\langle n| J_{\gamma}^{v}(x) J_{\gamma}^{\mu}(0)|n\rangle-2\langle n| J_{z}^{\nu}(x) J_{\gamma}^{\mu}(0)|n\rangle+2\langle n| J_{W}^{v}(x)\left(J_{W}^{+\mu}(0)\right)_{V}|n\rangle
\end{aligned}
$$

- Thus they be in principle related to ordinary DIS (pure $\gamma$-exchange), PVDIS DIS ( $\gamma$-W interference) and neutrino DIS (pure W-exchange) structure functions.
- At leading twist this should give identical result to Marciano-Sirlin at UV regime through sum rules of structure functions.


## Brief Summary

- Box diagram gives the largest theoretical uncertainty to the radiative corrections in beta decay
- Marciano + Sirlin: Current algebra+OPE. Intermediate state contribution needs modeling and the estimated uncertainty is 100\%
- A dispersion relation approach allows a direct mapping between the box diagram with experimental observables
- Several issues:

1. What are the available data, especially at moderate energy scale
2. How much modeling is needed, especially for $\mathrm{V}^{*} \mathrm{~A}$
3. How to systematically reconcile with and improve from Marciano-Sirlin result
