Virtual photon-photon scattering and $a_\mu$: dispersive approach to hadronic light-by-light

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Outline

Introduction: $(g - 2)_{\mu}$ and hadronic light-by-light

A dispersive approach to HLbL
   Introduction and main result
   Derivation of the Master Formula

Conclusions

arXiv:1402.7081
in collaboration with M. Hoferichter, M. Procura and P. Stoffer
Status of \((g-2)_{\mu}\), experiment vs SM

\[\begin{align*}
\text{HMNT (06)} \\
\text{JN (09)} \\
\text{Davier et al, } \tau \ (10) \\
\text{Davier et al, } e^+e^- \ (10) \\
\text{JS (11)} \\
\text{HLMNT (10)} \\
\text{HLMNT (11)} \\
\text{experiment} \\
\text{BNL} \\
\text{BNL (new from shift in } \lambda) \\
\end{align*}\]

\[a_{\mu} \times 10^{10} = 11659000\]
**Status of \((g - 2)_\mu\), experiment vs SM**

Different contributions to the total SM result

<table>
<thead>
<tr>
<th></th>
<th>(a_\mu) ([10^{-11}])</th>
<th>(\Delta a_\mu) ([10^{-11}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>experiment</td>
<td>116 592 089.</td>
<td>63.</td>
</tr>
<tr>
<td>QED (O(\alpha))</td>
<td>116 140 973.21</td>
<td>0.03</td>
</tr>
<tr>
<td>QED (O(\alpha^2))</td>
<td>413 217.63</td>
<td>0.01</td>
</tr>
<tr>
<td>QED (O(\alpha^3))</td>
<td>30 141.90</td>
<td>0.00</td>
</tr>
<tr>
<td>QED (O(\alpha^4))</td>
<td>381.01</td>
<td>0.02</td>
</tr>
<tr>
<td>QED (O(\alpha^5))</td>
<td>5.09</td>
<td>0.01</td>
</tr>
<tr>
<td>QED total</td>
<td>116 584 718.95</td>
<td>0.04</td>
</tr>
<tr>
<td>electroweak, total</td>
<td>153.6</td>
<td>1.0</td>
</tr>
</tbody>
</table>

| HVP (LO)            | 6 949.                    | 43.                      |
| HVP (HO)            | -98.                      | 1.                       |
| HLbL                | 116.                      | 40.                      |

theory 116 591 839. 59.
Hadronic light-by-light: irreducible uncertainty?

- Hadronic contributions responsible for most of the theory uncertainty

- Hadronic vacuum polarization (HVP) can be systematically improved
  (but going much below 1% is hard – dealing with radiative corrections poses serious problems)

- Hadronic light-by-light (HLbL) is more problematic:
  - “it cannot be expressed in terms of measurable quantities”
  - reliability of uncertainty estimate based more on consensus than on a systematic method
  - only first-principle method in sight: lattice QCD (when will it become competitive?)
Different evaluations of HLbL

Table 13
Summary of the most recent results for the various contributions to $a^\mu_{\text{HLbL\,had}} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

<table>
<thead>
<tr>
<th>Contribution</th>
<th>BPP</th>
<th>HKS</th>
<th>KN</th>
<th>MV</th>
<th>BP</th>
<th>PdRV</th>
<th>NJ/NJN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0, \eta, \eta'$</td>
<td>85 ± 13</td>
<td>82.7 ± 6.4</td>
<td>83 ± 12</td>
<td>114 ± 10</td>
<td>-</td>
<td>114 ± 13</td>
<td>99 ± 16</td>
</tr>
<tr>
<td>$\pi, K$ loops</td>
<td>-19 ± 13</td>
<td>-4.5 ± 8.1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-19 ± 19</td>
<td>-19 ± 13</td>
</tr>
<tr>
<td>$\pi, K$ loops + other subleading in $N_c$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0 ± 10</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Axial vectors</td>
<td>2.5 ± 1.0</td>
<td>1.7 ± 1.7</td>
<td>-</td>
<td>22 ± 5</td>
<td>-</td>
<td>15 ± 10</td>
<td>22 ± 5</td>
</tr>
<tr>
<td>Scalars</td>
<td>-6.8 ± 2.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-7 ± 7</td>
<td>-7 ± 2</td>
</tr>
<tr>
<td>Quark loops</td>
<td>21 ± 3</td>
<td>9.7 ± 11.1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2.3 ±</td>
<td>21 ± 3</td>
</tr>
<tr>
<td>Total</td>
<td>83 ± 32</td>
<td>89.6 ± 15.4</td>
<td>80 ± 40</td>
<td>136 ± 25</td>
<td>110 ± 40</td>
<td>105 ± 26</td>
<td>116 ± 39</td>
</tr>
</tbody>
</table>

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts ($K$s are subdominant)
- heavier single-particle poles decreasingly important (unless one models them to resum the high-energy tail)
Approaches to Hadronic light-by-light

- **Model calculations**
  - ENJL
  - NJL and hidden gauge
  - nonlocal $\chi$QM
  - AdS/CFT
  - Dyson-Schwinger
  - constituent $\chi$QM
  - resonances in the narrow-width limit

- **Impact of rigorously derived constraints**
  - high-energy constraints taken into account in several models above
    - addressed specifically by
    - high-energy constraints related to the axial anomaly
    - sum rules for $\gamma^* \gamma \rightarrow X$
  - low-energy constraints–pion polarizabilities

- **Lattice**
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Some notation

HLbL tensor:

\[ \Pi_{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz \ e^{-i(x \cdot q_1 + y \cdot q_2 + z \cdot q_3)} \langle 0 | T \{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0) \} | 0 \rangle \]

where

\[ j^\mu(x) = \sum_i Q_i \psi_i(x) \gamma^\mu q_i(x), \ i = u, d, s \]

\[ k = q_1 + q_2 + q_3 \quad k^2 = 0 \]

Helicity amplitudes

\[ H_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s, t, u) \equiv \mathcal{M}(\gamma^*(q_1, \lambda_1) \gamma^*(q_2, \lambda_2) \to \gamma^*(-q_3, \lambda_3) \gamma(k, \lambda_4)) \]

\[ = \epsilon_\mu(\lambda_1, q_1) \epsilon_\nu(\lambda_2, q_2) \epsilon^*_\lambda(\lambda_3, -q_3) \epsilon^*_\sigma(\lambda_4, k) \Pi_{\mu\nu\lambda\sigma} \]

with Mandelstam variables

\[ s = (q_1 + q_2)^2 = (k - q_3)^2 \quad t = (q_1 + q_3)^2 = (k - q_2)^2 \quad u = (q_2 + q_3)^2 = (k - q_1)^2 \]

and s-channel scattering angle

\[ z_s = \cos \theta_s = \frac{s}{(s - q_3^2) \sqrt{\lambda_{12}}} \left( t - u + \frac{(q_1^2 - q_2^2) q_3^2}{s} \right) \quad \lambda_{12} = \lambda(s, q_1^2, q_2^2) \]
Contribution to $a_\mu$

From gauge invariance:

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2) = -k^\rho \frac{\partial}{\partial k^\sigma} \Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2).$$

Contribution to $a_\mu$:

$$a_\mu = \lim_{k \to 0} \text{Tr} \left\{ (p + m) \Lambda^\rho (p', p) (p' + m) \Gamma_\rho (p', p) \right\}$$

$$\Gamma_\rho = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 q_3^2} \frac{\gamma^\mu (p' + q_1 + m) \gamma^\lambda (p - q_2 + m) \gamma^\nu}{((p' + q_1)^2 - m^2)((p - q_2)^2 - m^2)} k^\sigma \partial k^\rho \Pi_{\mu\nu\lambda\sigma}$$

with the projector

$$\Lambda^\rho (p', p) = \frac{m^2}{k^2 (4m^2 - k^2)} \left\{ \gamma^\rho + \frac{k^2 + 2m^2}{m(k^2 - 4m^2)} (p + p')^\rho \right\}$$

$m$ denotes the mass of the muon, $p$ and $p' = p - k$ the momenta of the incoming and outgoing muon, respectively
Setting up the dispersive calculation

We split the HLbL tensor as follows:

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots \]

Pion pole: known
Setting up the dispersive calculation

We split the HLbL tensor as follows:

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\text{0-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}} + \tilde{\Pi}_{\mu\nu\lambda\sigma} + \cdots \]

\[ F_{\pi}^V(q_1^2) F_{\pi}^V(q_2^2) F_{\pi}^V(q_3^2) \times \left[ \begin{array}{c}
\text{Contributions with two simultaneous cuts} \\
- \text{analytic properties like the box diagram in sQED} \\
- \text{triangle and bulb diagram required by gauge invariance} \\
- \text{multiplication with } F_{\pi}^V \text{ gives the correct } q^2 \text{ dependence} \\
\text{it is not an approximation!}
\end{array} \right] \]
Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu \nu \lambda \sigma} = \Pi_{\mu \nu \lambda \sigma}^{\pi^0\text{-pole}} + \Pi_{\mu \nu \lambda \sigma}^{\text{FsQED}} + \bar{\Pi}_{\mu \nu \lambda \sigma} + \cdots$$

The "rest" with $2\pi$ intermediate states has cuts only in one channel and is what will be calculated dispersively.
Setting up the dispersive calculation

We split the HLbL tensor as follows:

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi_0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \tilde{\Pi}_{\mu\nu\lambda\sigma} + \cdots \]

Contributions of cuts with anything else other than one and two pions in intermediate states will be neglected.
### Master formula

\[
\begin{align*}
    a_{\mu}^{\pi\pi} &= e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{l_{\pi\pi}}{q_1^2 q_2^2 s((p + q_1)^2 - m^2)((p - q_2)^2 - m^2)}, \\
l_{\pi\pi} &= \sum_{i \in \{1,2,3,6,14\}} \left( T_{i,s} I_{i,s} + 2 T_{i,u} I_{i,u} \right) + 2 T_{9,s} I_{9,s} + 2 T_{9,u} I_{9,u} + 2 T_{12,u} I_{12,u}
\end{align*}
\]

with \( I_{i,(s,u)} \) dispersive integrals and \( T_{i,(s,u)} \) integration kernels

\[
\begin{align*}
    I_{1,s} &= \frac{1}{\pi} \int \frac{ds'}{4m^2} \frac{d s'}{s' - s} \left( \frac{1}{s' - s} - \frac{s' - q_1^2 - q_2^2}{\lambda(s', q_1^2, q_2^2)} \right) \text{Im} \bar{h}_0^{++} (++(s'; q_1^2, q_2^2; s, 0)), \\
    T_{1,s} &= \frac{16}{3} s \left\{ m^2 + \frac{8P_{21} p \cdot q_1}{\lambda_{12}} \right\}, \quad T_{1,u} = \frac{16}{3} \left\{ \frac{4P_{12}^2}{\lambda_{12}} - P_{12} - Z_u \right\}
\end{align*}
\]
Master formula

\[
a_{\mu}^{\pi \pi} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{l_{\pi \pi}}{q_1^2 q_2^2 s((p + q_1)^2 - m^2)((p - q_2)^2 - m^2)},
\]

\[
l_{\pi \pi} = \sum_{i \in \{1,2,3,6,14\}} (T_i,s l_{i,s} + 2T_i,u l_{i,u}) + 2T_9,s l_{9,s} + 2T_9,u l_{9,u} + 2T_{12,u} l_{12,u}
\]

with \( l_{i,(s,u)} \) dispersive integrals and \( T_{i,(s,u)} \) integration kernels

\[
l_{1,s} = \frac{1}{\pi} \int_{\frac{4m^2}{s'}}^{\infty} \frac{ds'}{s' - s} \left( \frac{1}{s'} - \frac{s' - q_1^2 - q_2^2}{\lambda(s', q_1^2, q_2^2)} \right) \text{Im} \bar{h}_{0}^{0,++}(s'; q_1^2, q_2^2; s, 0),
\]

\[
l_{6,s} = \frac{1}{\pi} \int_{\frac{4m^2}{s'}}^{\infty} \frac{ds'}{(s' - q_1^2 - q_2^2)(s' - s)^2} \text{Im} \bar{h}_{2}^{2,+-}(s'; q_1^2, q_2^2; s, 0) \left( \frac{75}{8} \right)
\]

Helicity amplitudes contribute up to \( J = 2 \) (\( S \) and \( D \) waves)
The bars on the helicity amplitudes mean that we must subtract the FsQED contribution.

The unitarity relation for the barred imaginary parts read

\[
\text{Im}_s \bar{h}_{J,ij}(s) = \\
= h_{j,i}^c(s; q_1^2, q_2^2) \left( h_{j,j}^c(s; q_3^2, 0) \right)^* - N_{J,i}(s; q_1^2, q_2^2) N_{J,j}(s; q_3^2, 0) \\
+ \frac{1}{2} h_{j,i}^n(s; q_1^2, q_2^2) \left( h_{j,j}^n(s; q_3^2, 0) \right)^*
\]

where:

\( h_{J,i}^{c,n} \) = helicity amplitudes for \( \gamma^*\gamma^* \to \pi^+\pi^- \) and \( \pi^0\pi^0 \) resp.

\( N_{J,i} \) = partial-wave projection of the \( \gamma^*\gamma^* \to \pi^+\pi^- \) Born term
Master formula

What contributions are included? How?
Dispersions relations for $\gamma^* \gamma^* \rightarrow \pi\pi$

Roy-Steiner eqs. $=$ dispersion relations $+$ partial-wave expansion
$+$ crossing symmetry $+$ unitarity $+$ gauge invariance

- **On-shell** $\gamma \gamma \rightarrow \pi\pi$: prominent $D$-wave reson. $f_2(1270)$ Moussallam (10) Hoferichter, Phillips, Schat (11)

- $\gamma^* \gamma \rightarrow \pi\pi$ Moussallam (13)

- $\gamma^* \gamma^* \rightarrow \pi\pi$, new feature: anomalous thresholds Hoferichter, GC, Procura, Stoffer (13)
Dispersion relations for $\gamma^* \gamma^* \rightarrow \pi \pi$

Roy-Steiner eqs. = Dispersion relations + partial-wave expansion
+ crossing symmetry + unitarity + gauge invariance

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- $\gamma^* \gamma \rightarrow \pi \pi$ Moussallam (13)
- $\gamma^* \gamma^* \rightarrow \pi \pi$, new feature: **anomalous** thresholds Hoferichter, GC, Procura, Stoffer (13)

**Constraints**
- **Low energy**: pion polar., ChPT
- **Primakoff**: $\gamma \pi \rightarrow \gamma \pi$ at COMPASS, JLAB
- **Scattering**: $e^+ e^- \rightarrow e^+ e^- \pi \pi$, $e^+ e^- \rightarrow \pi \pi \gamma$
- **Decays**: $\omega, \phi \rightarrow \pi \pi \gamma$
Dispersion relations for $\gamma^*\gamma^* \rightarrow \pi\pi$

Roy-Steiner eqs. = Dispersion relations + partial-wave expansion + crossing symmetry + unitarity + gauge invariance

- **On-shell** $\gamma\gamma \rightarrow \pi\pi$: prominent $D$-wave reson. $f_2(1270)$ Moussallam (10) Hoferichter, Phillips, Schat (11)
- $\gamma^*\gamma \rightarrow \pi\pi$ Moussallam (13)
- $\gamma^*\gamma^* \rightarrow \pi\pi$, new feature: anomalous thresholds Hoferichter, GC, Procura, Stoffer (13)

Analysis of the Roy-Steiner equations for $\gamma^*\gamma^* \rightarrow \pi\pi$ is in progress: any experimental input most welcome
Physics of $\gamma^*\gamma^* \rightarrow \pi\pi$

- $\pi\pi$ rescattering $\Leftrightarrow$ resonances, e.g. $f_2(1270)$
- S-wave provides model-independent implementation of the $\sigma$
Physics of $\gamma^* \gamma^* \rightarrow \pi \pi$

- $\pi \pi$ rescattering $\Leftrightarrow$ resonances, e.g. $f_2(1270)$
- S-wave provides model-independent implementation of the $\sigma$
- Analytic continuation with dispersion theory: resonance properties
  - Precise determination of $\sigma$-pole from $\pi \pi$ scattering Caprini, GC, Leutwyler 2006

\[
M_\sigma = 441^{+16}_{-8} \text{ MeV} \quad \Gamma_\sigma = 544^{+18}_{-25} \text{ MeV}
\]

- Coupling $\sigma \rightarrow \gamma \gamma$ from $\gamma \gamma \rightarrow \pi \pi$
  Hoferichter, Phillips, Schat 2011

\[
\Gamma(\gamma \gamma) = \begin{array}{c}
1.7 \pm 0.4 \\
3.06 \pm 0.82 \\
2.08 \pm 0.2 \pm 0.07 \pm 0.04 \\
2.08 \\
1.2 \pm 0.4 \\
3.9 \pm 0.6 \\
1.8 \pm 0.4
\end{array}
\]

\[
\Gamma_2
\]

\[
\begin{array}{l}
\text{VALUE (keV)} \\
\text{DOCUMENT ID} \quad \text{TECN} \quad \text{COMMENT}
\end{array}
\]

\[
\begin{array}{l}
1.2 \pm 0.4 \quad \text{MOUSSALLAM11} \quad \text{RVUE} \quad \text{Compilation}
\end{array}
\]

\[
\begin{array}{l}
(445 \pm 25) - (278 \pm 18) \\
(457 \pm 14) - (279 \pm 11) \\
(44\pm 3) - (274 \pm 6) \\
(452 \pm 13) - (259 \pm 16) \\
(448 \pm 43) - (266 \pm 43) \\
(455 \pm 6 \pm 31) - (278 \pm 6 \pm 34) \\
1.2 \text{ GARCIA-MAR.11} \quad \text{RVUE} \quad \text{Compilation}
\end{array}
\]

Note that $\Gamma \approx 2 \text{ Im}(\sqrt{s})$
A reliable evaluation of the HLbL requires many different contributions by and a collaboration among theorists and experimentalists.
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A convenient basis

$\Pi^{\mu\nu\rho\sigma}$: gauge + Lorentz inv. + $(k^2 = 0) \Rightarrow 29$ scalar functions

But: in such a minimal basis crossing symmetry is hidden

A convenient (redundant) basis:

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \left( A_{i,s}^{\mu\nu\lambda\sigma} \Pi_i(s, t, u) + A_{i,t}^{\mu\nu\lambda\sigma} \Pi_i(t, s, u) + A_{i,u}^{\mu\nu\lambda\sigma} \Pi_i(u, t, s) \right)$$
A convenient basis

\( \Pi^{\mu\nu\rho\sigma} : \text{gauge} + \text{Lorentz inv.} + (k^2 = 0) \Rightarrow 29 \text{ scalar functions} \)

But: in such a minimal basis crossing symmetry is hidden

A convenient (redundant) basis:

\[
\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \left( A^{\mu\nu\lambda\sigma}_{i,s} \Pi_i(s, t, u) + A^{\mu\nu\lambda\sigma}_{i,t} \Pi_i(t, s, u) + A^{\mu\nu\lambda\sigma}_{i,u} \Pi_i(u, t, s) \right)
\]

where (just one example):

\[
A^{\mu\nu\lambda\sigma}_{1,s} = \frac{8}{(s - q_3^2)\lambda_{12}} \left( k^\lambda q_3^\sigma - k \cdot q_3 g^{\lambda\sigma} \right) \left( q_{12}^{\mu\nu} + \frac{\lambda_{12}}{4} g^{\mu\nu} \right)
\]

\( A^{\mu\nu\lambda\sigma}_{i,t} \) from \((q_2, \nu) \leftrightarrow (q_3, \lambda)\)

\( A^{\mu\nu\lambda\sigma}_{i,u} \) from \((q_1, \mu) \leftrightarrow (q_3, \lambda)\)

\( \Rightarrow \text{crossing symmetry is explicit} \)
A convenient basis

$\Pi^{\mu\nu\rho\sigma}$: gauge + Lorentz inv. + $(k^2 = 0) \Rightarrow 29$ scalar functions

**But**: in such a minimal basis crossing symmetry is hidden

A convenient (redundant) basis:

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \left( A^{\mu\nu\lambda\sigma}_{i,s} \Pi_i(s, t, u) + A^{\mu\nu\lambda\sigma}_{i,t} \Pi_i(t, s, u) + A^{\mu\nu\lambda\sigma}_{i,u} \Pi_i(u, t, s) \right)$$

**Essential property of this basis**: the helicity amplitudes in each channel are “diagonal”:

$$\bar{H}_{++,++}(s, t, u) = \Pi_1(s, t, u) + \hat{H}_{++,++}(s, t, u)$$

$$\bar{H}_{00,++}(s, t, u) = -\frac{q_1^2 q_2^2}{\xi_1 \xi_2} \Pi_2(s, t, u) + \hat{H}_{00,++}(s, t, u)$$
A convenient basis

$\Pi^{\mu\nu\rho\sigma}$: gauge + Lorentz inv. + $(k^2 = 0) \Rightarrow 29$ scalar functions  

**But:** in such a minimal basis crossing symmetry is hidden

A convenient (redundant) basis:

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \left( A^{\mu\nu\lambda\sigma}_{i,s} \Pi_i(s, t, u) + A^{\mu\nu\lambda\sigma}_{i,t} \Pi_i(t, s, u) + A^{\mu\nu\lambda\sigma}_{i,u} \Pi_i(u, t, s) \right)$$

**Essential property of this basis:** the helicity amplitudes in each channel are “diagonal” and unitarity relations “simple”:

$$\text{Im}_s \bar{H}_{++,++}(s, t, u) = \text{Im}_s \Pi_1(s, t, u)$$

$$\text{Im}_s \bar{H}_{00,++}(s, t, u) = -\frac{q_1^2 q_2^2}{\xi_1 \xi_2} \text{Im}_s \Pi_2(s, t, u)$$

The cut in the $s$-channel of each $s$-channel helicity amplitude is only due to **one single** $\Pi_i(s, t, u)$ function (which only has a cut in $s$)
Unitarity relations for helicity amplitudes

Helicity amplitudes admit a partial wave expansion

\[ H_{\lambda_1 \lambda_2, \lambda_3 \lambda_4}(s, t, u) = \sum_J D_J^J(z_s) h_J^{\lambda_1 \lambda_2, \lambda_3 \lambda_4}(s) \]

where \( D_J^J(z_s) \) is the appropriate Wigner function.

Each partial wave satisfies a simple unitarity relation (for \( s > 0 \))

\[ \text{Im} h_J^{\lambda_1 \lambda_2, \lambda_3 \lambda_4}(s) = \frac{\sigma s}{16 \pi} \theta(s - 4m_{\pi}^2) h_{J, \lambda_1 \lambda_2}(s; q_1^2, q_2^2) h_{J, \lambda_3 \lambda_4}^*(s; q_3^2, 0) \]

where \( h_{J, \lambda_1 \lambda_2}(s; q_1^2, q_2^2) \) are partial-wave helicity amplitudes of the subprocess \( \gamma^* \gamma^* \rightarrow \pi \pi \).
Dispersion relations for the $\Pi_i(s, t, u)$

- The $\Pi_i(s, t, u)$ only have a cut in $s$ and for $s \geq 4m^2_\pi$

- Their imaginary part coincides with that of the related helicity amplitude

- The latter can be expanded in partial waves and for each of them unitarity fixes the imaginary part in terms of partial-wave helicity amplitudes of the subprocess $\gamma^*\gamma^* \rightarrow \pi\pi$

- A dispersive integral over the right-hand cut of each partial wave would in principle allow me to reconstruct the whole $\Pi_i(s, t, u)$, up to a polynomial
Simplified dispersion relations for the $\Pi_i(s, t, u)$

We will carry out the program outlined in the previous slide with one simplification (but see later!):

for each $\Pi_i(s, t, u)$ we only keep the discontinuity due to the lowest partial wave (i.e. $S$ or $D$)

all $\Pi_i(s, t, u)$ become single-variable functions $\Rightarrow \Pi_i(s)$

This is analogous to what is done for $\pi\pi$ scattering, $\eta \to 3\pi$ and several other processes when the amplitude is expressed as a sum of single-variable functions having only a right-hand cut, and goes under the name of “reconstruction theorem”

Stern, Sazdjian, Fuchs (93)
Fixing subtraction constants: soft-photon zeros

Gauge-invariance implies the presence of so-called soft-photon zeros

\[ H_{\lambda_1\lambda_2,\lambda_3\lambda_4} \xrightarrow{k \to 0} \propto (s - q_3^2) \]

and analogously

\[ H_{\lambda_1\lambda_2,\lambda_3\lambda_4} \xrightarrow{q_{1,2} \to 0} \propto (s - q_{2,1}^2) \]

In a dispersive representation such a property must emerge from the kernels of the dispersive integrals

and constrains the subtraction polynomial
Soft-photon zeros in $\gamma^*\gamma^* \rightarrow \pi\pi$

These soft-photon zeros can be studied also in subprocess $\gamma^*\gamma^* \rightarrow \pi\pi$ where a dispersive representation for the helicity amplitudes reads

$$h_{J, i}(s) = \frac{1}{\pi} \sum_{J' \text{ even}} \sum_{j=1}^{5} \int_{4m^2}^{\infty} ds' K_{jj'}^{ij}(s, s') \text{Im} h_{J', j}(s') + \cdots, \quad i, j \in \{\lambda_1 \lambda_2\}$$

the ellipsis stands for integrals of crossed-channel partial waves

The diagonal kernel functions

$$K_{00}^{++},[++](t, t') = K_{00}^{00,00}(t, t') = \frac{1}{t' - t} - \frac{t' - q_1^2 - q_2^2}{\lambda(t', q_1^2, q_2^2)}$$
$$K_{22}^{++},[++](t, t') = K_{22}^{00,00}(t, t') = \frac{p_t^2 q_t^2}{p_t'^2 q_t'^2} \left( \frac{1}{t' - t} - \frac{t' - q_1^2 - q_2^2}{\lambda(t', q_1^2, q_2^2)} \right)$$

display the desired soft-photon behaviour
Soft-photon zeros in $\gamma^*\gamma^* \rightarrow \pi\pi$

Soft-photon zeros of the $\gamma^*\gamma^* \rightarrow \pi\pi$ sub-amplitudes manifest themselves as a modification of the Cauchy kernel by a factor:

$$K_{12}(s, s') = \frac{f_{12}(s, s')}{s' - s}, \quad K_{34}(s, s') = \frac{f_{34}(s, s')}{s' - s},$$

for the initial- and final-state photon pair, respectively.

A modified Cauchy kernel that gives the HLbL tensor the proper soft-photon zeros is obtained by factorization

$$K_{12,34}(s, s') = \frac{f_{12}(s, s')f_{34}(s, s')}{s' - s}.$$
Dispersion relations for the $\Pi_i(s)$

Imposing the same form of the soft-photon zeros as in the subamplitudes $\gamma^*\gamma^* \to \pi\pi$ we obtain the following dispersion relations:

$$\Pi_1^s = \tilde{h}^0_{++,++}(s) = \frac{s - q_3^2}{\pi} \int_{4m_{\pi}^2}^{\infty} \frac{ds'}{s' - q_3^2} \left( \frac{1}{s' - s} - \frac{s' - q_1^2 - q_2^2}{\lambda'_{12}} \right) \text{Im} \tilde{h}^0_{++,++}(s')$$

$$y\Pi_2^s = \tilde{h}^0_{00,++}(s) = \frac{s - q_3^2}{\pi} \int_{4m_{\pi}^2}^{\infty} \frac{ds'}{s' - q_3^2} \left( \frac{1}{s' - s} - \frac{s' - q_1^2 - q_2^2}{\lambda'_{12}} \right) \text{Im} \tilde{h}^0_{00,++}(s')$$

with $y = -\frac{q_1^2 q_2^2}{\xi_1 \xi_2}$ [and similarly for the others]
Dispersion relations for the $\Pi_i(s)$

Soft-photon zeros for the $\Pi_i^s$ or for the helicity amplitudes?

Remember

$$\bar{H}_{++,++}(s, t, u) = \Pi_i^s + \hat{H}_{++,++}(s, t, u)$$

with

$$\hat{H}_{\lambda_1 \lambda_2, \lambda_3 \lambda_4}(s, t, u) = \sum_{i=1}^{15} \left( f_{\lambda_1 \lambda_2, \lambda_3 \lambda_4}^i \Pi_i^t + \tilde{f}_{\lambda_1 \lambda_2, \lambda_3 \lambda_4}^i \Pi_i^u \right)$$
Dispersion relations for the $\Pi_i(s)$

Soft-photon zeros for the $\Pi_i^s$ or for the helicity amplitudes?

Remember

$$\tilde{H}_{++,++}(s, t, u) = \Pi_i^s + \hat{H}_{++,++}(s, t, u)$$

with

$$\hat{H}_{\lambda_1 \lambda_2, \lambda_3 \lambda_4}(s, t, u) = \sum_{i=1}^{15} \left( f^i_{\lambda_1 \lambda_2, \lambda_3 \lambda_4} \Pi_i^t + \tilde{f}^i_{\lambda_1 \lambda_2, \lambda_3 \lambda_4} \Pi_i^u \right)$$

By sheer kinematics the soft-photon zeros imposed on the $\Pi_i^s$ imply the correct soft-photon zeros to the full helicity amplitudes
Our dispersive representation of the HLbL tensor

\[ \bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \left( A^{\mu\nu\lambda\sigma}_{i,s} \Pi_i(s) + A^{\mu\nu\lambda\sigma}_{i,t} \Pi_i(t) + A^{\mu\nu\lambda\sigma}_{i,u} \Pi_i(u) \right) \]

- the \( \Pi_i(s) \) are single-variable functions having only a right-hand cut
- for the discontinuity we keep only the lowest partial wave
- the dispersive integral that gives the \( \Pi_i(s) \) in terms of its discontinuity has the required soft-photon zeros
- soft-photon zeros constrain the subtraction polynomial to vanish (unless one wanted to subtract more, which is unnecessary)
Contribution of $\bar{\Pi}_{\mu \nu \lambda \sigma}$ to $a_\mu$

$$a_\mu = \lim_{k \to 0} \text{Tr} \left\{ (p + m) \gamma^\rho (p', p) (p' + m) \Gamma_\rho (p', p) \right\}$$

$$\Gamma_\rho = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 q_3^2} \frac{\gamma^\mu (p' + q_1 + m) \gamma^\lambda (p - q_2 + m) \gamma^\nu k^\sigma \partial_\kappa \Pi_{\mu \nu \lambda \sigma}}{((p' + q_1)^2 - m^2)((p - q_2)^2 - m^2)}$$

A technical caveat: a disadvantage of the basis we chose is that the helicity amplitudes have kinematical singularities – the full HLbL tensor, however, doesn’t.

⇒ In order to make sense of the limit $k_\mu \to 0$ for $\bar{\Pi}_{\mu \nu \lambda \sigma}$ we must average over the direction of $k_\mu$ first
Contribution of $\tilde{\Pi}_{\mu\nu\lambda\sigma}$ to $a_\mu$

$$a_\mu = \frac{1}{16m} \text{Tr}\left\{ (\not{p} + m) [\gamma^\rho, \gamma^\tau] (\not{p} + m) \tilde{\Gamma}_{\rho\tau} \right\}$$

$$\tilde{\Gamma}_{\rho\tau} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 s} \frac{\gamma^\mu (\not{p} + q_1 + m) \gamma^\lambda (\not{p} - q_2 + m) \gamma^\nu}{((p + q_1)^2 - m^2)((p - q_2)^2 - m^2)}$$

$$\times \left[ \int \frac{d\Omega(p, k)}{4\pi} \frac{k_\tau}{k^2} \frac{\partial}{\partial k^\rho} \tilde{\Pi}_{\mu\nu\lambda\sigma} \right]_{k=0}$$
Contribution of $\widetilde{\Pi}_{\mu\nu\lambda\sigma}$ to $a_\mu$

$$a_\mu = \frac{1}{16m} \text{Tr}\left\{ (\not{p} + m) [\gamma^\rho, \gamma^\tau] (\not{p} + m) \tilde{\Gamma}_\rho \tau \right\}$$

$$\tilde{\Gamma}_\rho \tau = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 s} \frac{\gamma^\mu (\not{p} + q_1 + m) \gamma^\lambda (\not{p} - q_2 + m) \gamma^\nu}{((p + q_1)^2 - m^2)((p - q_2)^2 - m^2)}$$

$$\times \left[ \int \frac{d\Omega(p, k)}{4\pi} k_\tau k^\sigma \frac{\partial}{\partial k^\rho} \tilde{\Pi}_{\mu\nu\lambda\sigma} \right]_{k=0}$$

- all $A^i_{\mu\nu\rho\sigma}$ tensors scale like $O(k^0)$
- any term of $O(k^2)$ in the $\Pi_i(s)$ does not contribute to $a_\mu$
- higher partial waves in $\Pi_i(s)$ are suppressed by angular momentum factors:

$$q_{34}^2 = (s - q_3^2)^2/(4s) = O(k^2)$$

- $\Rightarrow$ keeping only the lowest partial wave in the discontinuity of the $\Pi_i(s)$ is not an approximation for the calculation of $a_\mu$
Master formula

\[
a^\pi\pi_\mu = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{I^{\pi\pi}}{q_1^2 q_2^2 s((p + q_1)^2 - m^2)((p - q_2)^2 - m^2)},
\]

\[
I^{\pi\pi} = \sum_{i \in \{1, 2, 3, 6, 14\}} \left( T_{i,s} I_{i,s} + 2 T_{i,u} I_{i,u} \right) + 2 T_{9,s} I_{9,s} + 2 T_{9,u} I_{9,u} + 2 T_{12,u} I_{12,u}
\]

with \(I_{i,(s,u)}\) dispersive integrals and \(T_{i,(s,u)}\) integration kernels
Conclusions and outlook

- I have presented a dispersive framework for the calculation of the HLbL contribution to $a_\mu$

- which takes into account only single- and double-pion intermediate states
  
The extension to other single-particle intermediate states ($\eta, \eta'$, etc.) is trivial

- we have derived a master formula which expresses the contribution of $2\pi$ intermediate states to $a_\mu$ in terms of (integrals over) $\gamma^*\gamma^* \rightarrow \pi\pi$ partial waves

- a numerical evaluation of the master formula is in progress

- we believe that this is a step towards a model-independent calculation of the HLbL contribution to $a_\mu$
A reliable evaluation of the HLbL requires many different contributions by and a collaboration among theorists and experimentalists.
SM contributions to \((g - 2)_\mu\): QED

\[
\begin{align*}
\alpha^\text{QED}_\mu &= (1/2)(\alpha/\pi) \\
&\quad + 0.765857426 \times 16 \times (\alpha/\pi)^2 \\
&\quad + 24.05050988 \times 28 \times (\alpha/\pi)^3 \\
&\quad + 130.8796 \times 63 \times (\alpha/\pi)^4 \\
&\quad + 753.29 \times 1.04 \times (\alpha/\pi)^5 \quad \text{COMPLETED!}
\end{align*}
\]

Adding up, we get:

\[
\alpha^\text{QED}_\mu = 116584718.951 \times 10^{-11}
\]

from coeffs, mainly from 4-loop unc

with \(\alpha = 1/137.035999049(90) \) [0.66 ppb]

Slide by Massimo Passera
SM contributions to $(g - 2)_\mu$: electroweak

**One-loop term:**

$$a_{\mu}^{\text{EW}}(1\text{-loop}) = \frac{5G_F m_{\mu}^2}{24\sqrt{2}\pi^2} \left[1 + \frac{1}{5} \left(1 - 4 \sin^2 \theta_W\right)^2 + O\left(\frac{m_{\mu}^2}{M_{Z,W,H}^2}\right)\right] \approx 195 \times 10^{-11}$$

1972: Jackiw, Weinberg; Bars, Yoshimura; Altarelli, Cabibbo, Maiani; Bardeen, Gastmans, Lautrup; Fujikawa, Lee, Sanda; Studenikin et al. '80s

**One-loop plus higher-order terms:**

$$a_{\mu}^{\text{EW}} = 153.6 (1) \times 10^{-11}$$

with $M_{\text{Higgs}} = 125.6 (1.5) \text{ GeV}$

Hadronic loop uncertainties and 3-loop nonleading logs.

Kukhto et al. '92; Czarnecki, Krause, Marciano '95; Knecht, Peris, Perrotet, de Rafael '02; Czarnecki, Marciano and Vainshtein '02; Degrassi and Giudice '99; Heinemeyer, Stockinger, Weiglein '04; Gribouk and Czarnecki '05; Vainshtein '03; Gnendiger, Stockinger, Stockinger-Kim 2013.

Slide by Massimo Passera
SM contributions to \((g - 2)_\mu\): HVP

\[ K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)s/m_\mu^2} \]

\[ a_\mu^{\text{HLO}} = \frac{1}{4\pi^3} \int_{4m_\pi^2}^\infty ds \ K(s) \sigma^{(0)}(s) = \frac{\alpha^2}{3\pi^2} \int_{4m_\pi^2}^\infty \frac{ds}{s} K(s) R(s) \]

\[ a_\mu^{\text{HLO}} = 6903 (53)_{\text{tot}} \times 10^{-11} \]

\[ = 6923 (42)_{\text{tot}} \times 10^{-11} \]

\[ = 6949 (37)_{\text{exp}} (21)_{\text{rad}} \times 10^{-11} \]

F. Jegerlehner and A. Nyffeler, Phys. Rept. 477 (2009) 1

F. Jegerlehner, A. Nyffeler, Phys. Rept. 477 (2009) 1

Davier et al, EPJ C71 (2011) 1515 (incl. BaBar & KLOE10 2n)

Hagiwara et al, JPG 38 (2011) 085003

Slide by Massimo Passera
SM contributions to \((g - 2)_{\mu}\): Higher-order HVP

- **HHO: Vacuum Polarization**

O(\(\alpha^3\)) contributions of diagrams containing hadronic vacuum polarization insertions:

\[
a_{\mu}^{HHO}(vp) = -98 (1) \times 10^{-11}
\]

Krause '96, Alemany et al. '98, Hagiwara et al. 2011

Only tiny shifts if \(\tau\) data are used instead of the \(e^+e^-\) ones

Davier & Marciano '04.
SM contributions to \((g - 2)_{\mu}:\) hadronic light-by-light

- **HHO: Light-by-light contribution**

  Unlike the HLO term, for the hadronic l-b-l term we must rely on theoretical approaches.

  This term had a troubled life! Latest values:

  \[
  \begin{align*}
  a_{\mu}^{\text{HHO}(\text{lbl})} &= +80 \pm 40 \times 10^{-11} \quad \text{Knecht & Nyffeler '02} \\
  a_{\mu}^{\text{HHO}(\text{lbl})} &= +136 \pm 25 \times 10^{-11} \quad \text{Melnikov & Vainshtein '03} \\
  a_{\mu}^{\text{HHO}(\text{lbl})} &= +105 \pm 26 \times 10^{-11} \quad \text{Prades, de Rafael, Vainshtein '09} \\
  a_{\mu}^{\text{HHO}(\text{lbl})} &= +116 \pm 39 \times 10^{-11} \quad \text{Jegerlehner & Nyffeler '09}
  \end{align*}
  \]

  Results based also on Hayakawa, Kinoshita '98 & '02; Bijnens, Pallante, Prades '96 & '02

  ‘Bound’ \(a_{\mu}^{\text{HHO}(\text{lbl})} < \sim 160 \times 10^{-11}\) Erler, Sanchez '06, Pivovarov '02; also Boughezal, Melnikov '11

  Lattice? Very hard... in progress. M. Golterman @ PhilPsi 2013; T. Blum @ Lattice 2012

  Pion exch. in holographic QCD agrees. D.K. Hong, D.Kim '09; Cappellio, Catà, D'Ambrosio '11

  “By far not complete” calculation: \(188 \times 10^{-11}\) Fischer et al, PRD87(2013)034013

  Had lbl is likely to become the ultimate limitation of the SM prediction.
SM contributions to $\mu$: $a_{\mu,\text{EXP}} = 116592089\, (63) \times 10^{-11}$

<table>
<thead>
<tr>
<th>$a_{\mu,\text{SM}} \times 10^{11}$</th>
<th>$\Delta a_{\mu} = a_{\mu,\text{EXP}} - a_{\mu,\text{SM}}$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>116 591 793 (66)</td>
<td>296 (91) $\times 10^{-11}$</td>
<td>3.2 [1]</td>
</tr>
<tr>
<td>116 591 813 (57)</td>
<td>276 (85) $\times 10^{-11}$</td>
<td>3.2 [2]</td>
</tr>
<tr>
<td>116 591 839 (58)</td>
<td>250 (86) $\times 10^{-11}$</td>
<td>2.9 [3]</td>
</tr>
</tbody>
</table>

with the “conservative” $a_{\mu,\text{HHO(lbl)}} = 116\, (39) \times 10^{-11}$ and the LO hadronic from: