Virtual photon-photon scattering and a_μ: dispersive approach to hadronic light-by-light

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Workshop Hadronic Probes of Fundamental Symmetries
Amherst Center for Fundamental interactions
Amherst, March 6-8, 2014

Outline

Introduction: $(g-2)_\mu$ and hadronic light-by-light

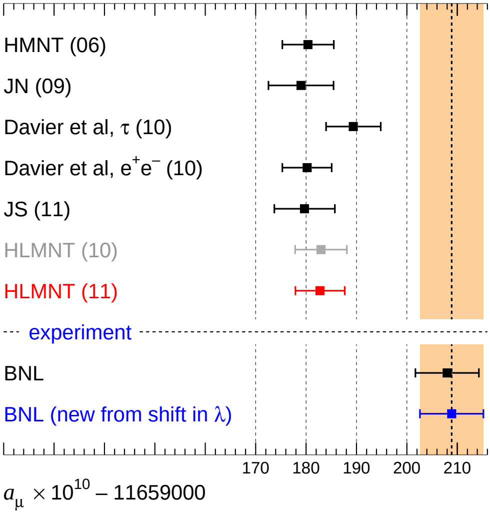
A dispersive approach to HLbL
- Introduction and main result
- Derivation of the Master Formula

Conclusions

arXiv:1402.7081

in collaboration with M. Hoferichter, M. Procura and P. Stoffer

Status of $(g-2)_\mu$, experiment vs SM



Hagiwara et al. 2012

Status of $(g-2)_\mu$, experiment vs SM

Different contributions to the total SM result

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.95	0.04
electroweak, total	153.6	1.0
HVP (LO) [Hagiwara et al. 2011]	6 949.	43.
HVP (HO) [Hagiwara et al. 2011]	−98.	1.
HLbL [Jegerlehner-Nyffeler 2009]	116.	40.
theory	116 591 839.	59.

Hadronic light-by-light: irreducible uncertainty?

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved
  (but going much below 1% is hard – dealing with radiative corrections poses serious problems)
- Hadronic light-by-light (HLbL) is more problematic:
  - "it *cannot* be expressed in terms of measurable quantities"
  - reliability of uncertainty estimate based more on consensus than on a systematic method
  - only first-principle method in sight: lattice QCD
    (when will it become competitive?)

Different evaluations of HLbL

Jegerlehner Nyffeler 2009

Table 13

Summary of the most recent results for the various contributions to $a_\mu^{\rm LbL;had} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	–	–	–	0 ± 10	–	–	–
Axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
Scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
Quark loops	21 ± 3	9.7 ± 11.1	–	–	–	$2.3\pm$	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (Ks are subdominant)
- heavier single-particle poles decreasingly important (unless one models them to resum the high-energy tail)

Approaches to Hadronic light-by-light

- Model calculations
  - ENJL — Bijnens, Pallante, Prades (95-96)
  - NJL and hidden gauge — Hayakawa, Kinoshita, Sanda (95-96)
  - nonlocal χQM — Dorokhov, Broniowski (08)
  - AdS/CFT — Cappiello, Cata, D'Ambrosio (10)
  - Dyson-Schwinger — Goecke, Fischer, Williams (11)
  - constituent χQM — Greynat, de Rafael (12)
  - resonances in the narrow-width limit — Pauk, Vanderhaeghen (14)
- Impact of rigorously derived constraints
  - high-energy constraints taken into account in several models above
    addressed specifically by — Knecht, Nyffeler (01)
  - high-energy constraints related to the axial anomaly — Melnikov, Vainshtein (04) and Nyffeler (09)
  - sum rules for $\gamma^*\gamma \to X$ — Pascalutsa, Pauk, Vanderhaeghen (12)
    see also: workshop MesonNet (13)
  - low-energy constraints–pion polarizabilities — Engel, Ramsey-Musolf (13)
- Lattice — Blum et al. (05,12)

Outline

Some notation

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz\, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0 | T\{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\} |0\rangle$$

where $j^\mu(x) = \sum_i Q_i \bar{q}_i(x)\gamma^\mu q_i(x)$, $i = u, d, s$

$$k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

Helicity amplitudes

$$\begin{aligned} H_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s,t,u) &\equiv \mathcal{M}(\gamma^*(q_1,\lambda_1)\gamma^*(q_2,\lambda_2) \to \gamma^*(-q_3,\lambda_3)\gamma(k,\lambda_4)) \\ &= \epsilon_\mu(\lambda_1,q_1)\epsilon_\nu(\lambda_2,q_2)\epsilon^*_\lambda(\lambda_3,-q_3)\epsilon^*_\sigma(\lambda_4,k)\Pi^{\mu\nu\lambda\sigma} \end{aligned}$$

with Mandelstam variables

$$s = (q_1+q_2)^2 = (k-q_3)^2 \quad t = (q_1+q_3)^2 = (k-q_2)^2 \quad u = (q_2+q_3)^2 = (k-q_1)^2$$

and s-channel scattering angle

$$z_s = \cos\theta_s = \frac{s}{(s-q_3^2)\sqrt{\lambda_{12}}}\left(t - u + \frac{(q_1^2 - q_2^2)q_3^2}{s}\right) \qquad \lambda_{12} = \lambda(s, q_1^2, q_2^2)$$

Contribution to a_μ

From gauge invariance:

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2) = -k^\rho \frac{\partial}{\partial k^\sigma} \Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2).$$

Contribution to a_μ:

$$a_\mu = \lim_{k \to 0} \mathrm{Tr}\Big\{ (\not{p} + m) \Lambda^\rho(p', p) (\not{p}' + m) \Gamma_\rho(p', p) \Big\}$$

$$\Gamma_\rho = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 q_3^2} \frac{\gamma^\mu (\not{p}' + \not{q}_1 + m) \gamma^\lambda (\not{p} - \not{q}_2 + m) \gamma^\nu}{\left((p' + q_1)^2 - m^2\right)\left((p - q_2)^2 - m^2\right)} k^\sigma \partial_{k^\rho} \Pi_{\mu\nu\lambda\sigma}$$

with the projector

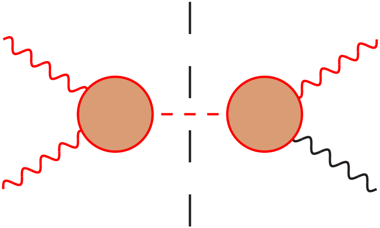
$$\Lambda^\rho(p', p) = \frac{m^2}{k^2(4m^2 - k^2)} \left\{ \gamma^\rho + \frac{k^2 + 2m^2}{m(k^2 - 4m^2)} (p + p')^\rho \right\}$$

m denotes the mass of the muon, p and $p' = p - k$ the momenta of the incoming and outgoing muon, respectively

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: known

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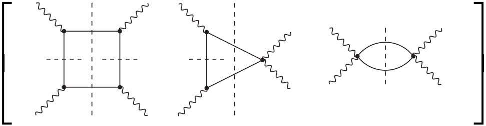
$$F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2) \times$$



Contribution with two simultaneous cuts

– analytic properties like the box diagram in sQED

– triangle and bulb diagram required by gauge invariance

– multiplication with F_π^V gives the correct q^2 dependence

it is not an approximation!

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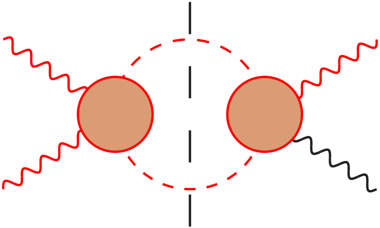

The "rest" with 2π intermediate states has cuts only in one channel and is what will be calculated dispersively

Setting up the dispersive calculation

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Contributions of cuts with anything else other than one and two pions in intermediate states will be neglected

Master formula

$$a_\mu^{\pi\pi} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{I^{\pi\pi}}{q_1^2 q_2^2 s\left((p+q_1)^2 - m^2\right)\left((p-q_2)^2 - m^2\right)},$$

$$I^{\pi\pi} = \sum_{i\in\{1,2,3,6,14\}} \left(T_{i,s} I_{i,s} + 2T_{i,u} I_{i,u}\right) + 2T_{9,s} I_{9,s} + 2T_{9,u} I_{9,u} + 2T_{12,u} I_{12,u}$$

with $I_{i,(s,u)}$ dispersive integrals and $T_{i,(s,u)}$ integration kernels

$$I_{1,s} = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'-s} \left(\frac{1}{s'-s} - \frac{s' - q_1^2 - q_2^2}{\lambda(s', q_1^2, q_2^2)} \right) \mathrm{Im}\bar{h}^0_{++,++}(s'; q_1^2, q_2^2; s, 0),$$

$$T_{1,s} = \frac{16}{3} s \left\{ m^2 + \frac{8 P_{21}\, p \cdot q_1}{\lambda_{12}} \right\}, \qquad T_{1,u} = \frac{16}{3} \left\{ \frac{4P_{12}^2}{\lambda_{12}} - P_{12} - Z_u \right\},$$

Master formula

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$$I_{6,s} = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{\left(s'-q_1^2-q_2^2\right)\left(s'-s\right)^2} \mathrm{Im}\bar{h}^2_{+-,+-}(s';q_1^2,q_2^2;s,0) \left(\frac{75}{8}\right)$$

Helicity amplitudes contribute up to $J=2$ (S and D waves)

Master formula

The bars on the helicity amplitudes mean that we must subtract the FsQED contribution.

The unitarity relation for the barred imaginary parts read

$$
\begin{aligned}
&\mathrm{Im}_s \bar{h}_{J,ij}(s) = \\
&= h^{\mathrm{c}}_{J,i}(s;q_1^2,q_2^2)\left(h^{\mathrm{c}}_{J,j}(s;q_3^2,0)\right)^* - N_{J,i}(s;q_1^2,q_2^2)N_{J,j}(s;q_3^2,0) \\
&+ \frac{1}{2}h^{\mathrm{n}}_{J,i}(s;q_1^2,q_2^2)\left(h^{\mathrm{n}}_{J,j}(s;q_3^2,0)\right)^*
\end{aligned}
$$

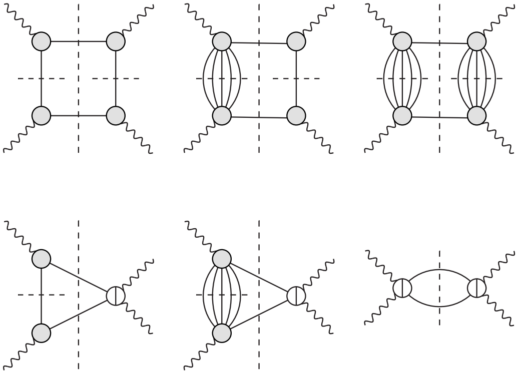
where:

$h^{\mathrm{c,n}}_{J,i}$ = helicity amplitudes for $\gamma^*\gamma^* \to \pi^+\pi^-$ and $\pi^0\pi^0$ resp.

$N_{J,i}$ = partial-wave projection of the $\gamma^*\gamma^* \to \pi^+\pi^-$ Born term

Master formula

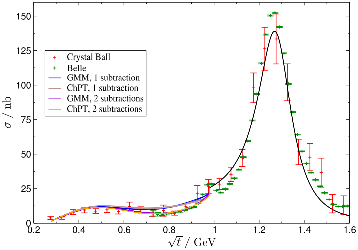
What contributions are included? How?



Dispersion relations for $\gamma^*\gamma^* \to \pi\pi$

Roy-Steiner eqs. $=$ Dispersion relations $+$ partial-wave expansion $+$ crossing symmetry $+$ unitarity $+$ gauge invariance

- On-shell $\gamma\gamma \to \pi\pi$: prominent *D*-wave reson. $f_2(1270)$ Moussallam (10) Hoferichter, Phillips, Schat (11)
- $\gamma^*\gamma \to \pi\pi$ Moussallam (13)
- $\gamma^*\gamma^* \to \pi\pi$, new feature: anomalous thresholds Hoferichter, GC, Procura, Stoffer (13)



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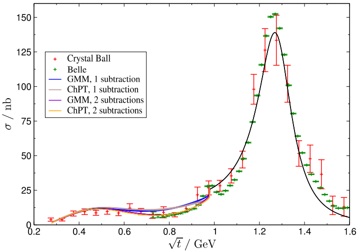
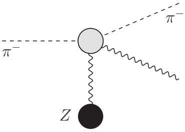
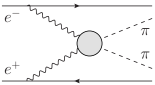
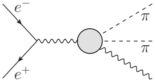


- Constraints
  - Low energy: pion polar., ChPT
  - Primakoff: $\gamma\pi \to \gamma\pi$ at COMPASS, JLAB
  - Scattering: $e^+e^- \to e^+e^-\pi\pi$, $e^+e^- \to \pi\pi\gamma$
  - Decays: $\omega, \phi \to \pi\pi\gamma$







Dispersion relations for $\gamma^*\gamma^* \to \pi\pi$

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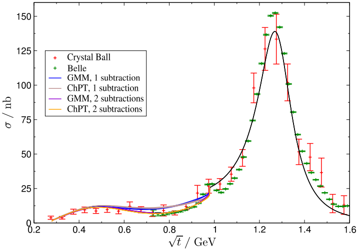
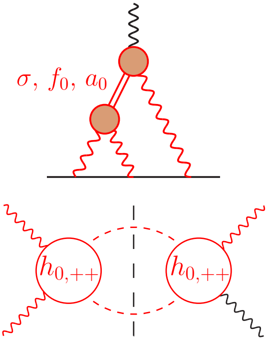


Analysis of the Roy-Steiner equations for $\gamma^*\gamma^* \to \pi\pi$ is in progress: any experimental input most welcome

Physics of $\gamma^*\gamma^* \to \pi\pi$

- $\pi\pi$ rescattering $\Leftrightarrow$ resonances, e.g. $f_2(1270)$
- *S*-wave provides model-independent implementation of the σ

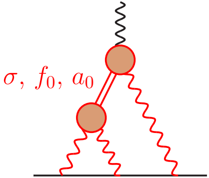


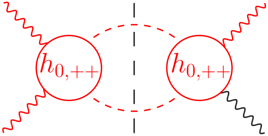
Physics of $\gamma^*\gamma^* \to \pi\pi$

- $\pi\pi$ rescattering $\Leftrightarrow$ resonances, e.g. $f_2(1270)$
- S-wave provides model-independent implementation of the σ
- Analytic continuation with dispersion theory: resonance properties
  - Precise determination of σ-pole from $\pi\pi$ scattering Caprini, GC, Leutwyler 2006

$$M_\sigma = 441^{+16}_{-8}\,\mathrm{MeV} \qquad \Gamma_\sigma = 544^{+18}_{-25}\,\mathrm{MeV}$$

  - Coupling $\sigma \to \gamma\gamma$ from $\gamma\gamma \to \pi\pi$ Hoferichter, Phillips, Schat 2011





$f_0(500)$ or σ
was $f_0(600)$

$I^G(J^{PC}) = 0^+(0^{++})$

A REVIEW GOES HERE – Check our WWW List of Reviews

$f_0(500)$ PARTIAL WIDTHS

$\Gamma(\gamma\gamma)$ Γ_2

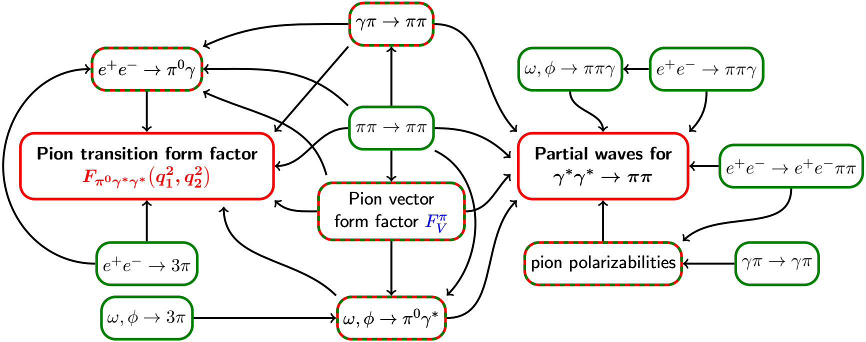
VALUE (keV)	DOCUMENT ID	TECN	COMMENT
• • • We do not use the following data for averages, fits, limits, etc. • • •			
1.7 ±0.4	[54] HOFERICHTER 11	RVUE	Compilation
3.08±0.82	[55] MENNESSIER 11	RVUE	Compilation
2.08±0.2 $^{+0.07}_{-0.04}$	[56] MOUSSALLAM 11	RVUE	Compilation
2.08	[57] MAO 09	RVUE	Compilation
1.2 ±0.4	[58] BERNABEU 08	RVUE	
3.9 ±0.6	[55] MENNESSIER 08	RVUE	$\gamma\gamma \to \pi^+\pi^-, \pi^0\pi^0$
1.8 ±0.4	[59] OLLER 08	RVUE	Compilation

$f_0(500)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\,\mathrm{Im}(\sqrt{s_{\mathrm{pole}}})$.

VALUE (MeV)	DOCUMENT ID	TECN	COMMENT
(400–550)$-i$(200–350) OUR ESTIMATE			
• • • We do not use the following data for averages, fits, limits, etc. • • •			
$(445 \pm 25)-i(278^{+22}_{-18})$	[1,2] GARCIA-MAR..11	RVUE	Compilation
$(457^{+14}_{-13})-i(279^{+11}_{-7})$	[1,3] GARCIA-MAR..11	RVUE	Compilation
$(442^{+5}_{-8})-i(274^{+6}_{-5})$	[4] MOUSSALLAM 11	RVUE	Compilation
$(452 \pm 13)-i(259 \pm 16)$	[5] MENNESSIER 10	RVUE	Compilation
$(448 \pm 43)-i(266 \pm 43)$	[6] MENNESSIER 10	RVUE	Compilation
$(455 \pm 6^{+31}\ldots)-i(278 \pm 6^{+34}\ldots)$	[7] CAPRINI 08	RVUE	Compilation

Hadronic light-by-light: a roadmap



Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among theorists and experimentalists

Outline

Introduction: $(g-2)_\mu$ and hadronic light-by-light

A dispersive approach to HLbL
- Introduction and main result
- Derivation of the Master Formula

Conclusions

A convenient basis

$\Pi^{\mu\nu\rho\sigma}$: gauge + Lorentz inv. + $(k^2 = 0) \Rightarrow$ 29 scalar functions
But: in such a minimal basis crossing symmetry is hidden

A convenient (redundant) basis:

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \Big(A^{\mu\nu\lambda\sigma}_{i,s} \Pi_i(s,t,u) + A^{\mu\nu\lambda\sigma}_{i,t} \Pi_i(t,s,u) + A^{\mu\nu\lambda\sigma}_{i,u} \Pi_i(u,t,s) \Big)$$

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where (just one example):

$$A_{1,s}^{\mu\nu\lambda\sigma} = \frac{8}{(s-q_3^2)\lambda_{12}} \left(k^\lambda q_3^\sigma - k\cdot q_3\, g^{\lambda\sigma} \right) \left(q_{12}^{\mu\nu} + \frac{\lambda_{12}}{4} g^{\mu\nu} \right)$$

$A_{i,t}^{\mu\nu\lambda\sigma}$ from $(q_2,\nu) \leftrightarrow (q_3,\lambda)$ $A_{i,u}^{\mu\nu\lambda\sigma}$ from $(q_1,\mu) \leftrightarrow (q_3,\lambda)$

$\Rightarrow$ crossing symmetry is explicit

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Essential property of this basis: the helicity amplitudes in each channel are "diagonal" :

$$\bar{H}_{++,++}(s,t,u) = \Pi_1(s,t,u) + \hat{H}_{++,++}(s,t,u)$$

$$\bar{H}_{00,++}(s,t,u) = -\frac{q_1^2 q_2^2}{\xi_1 \xi_2} \Pi_2(s,t,u) + \hat{H}_{00,++}(s,t,u)$$

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A convenient (redundant) basis:

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \left(A_{i,s}^{\mu\nu\lambda\sigma} \Pi_i(s,t,u) + A_{i,t}^{\mu\nu\lambda\sigma} \Pi_i(t,s,u) + A_{i,u}^{\mu\nu\lambda\sigma} \Pi_i(u,t,s) \right)$$

Essential property of this basis: the helicity amplitudes in each channel are "diagonal" and unitarity relations "simple":

$$\mathrm{Im}_s \bar{H}_{++,++}(s,t,u) = \mathrm{Im}_s \Pi_1(s,t,u)$$

$$\mathrm{Im}_s \bar{H}_{00,++}(s,t,u) = -\frac{q_1^2 q_2^2}{\xi_1 \xi_2} \mathrm{Im}_s \Pi_2(s,t,u)$$

The cut in the s-channel of each s-channel helicity amplitude is only due to one single $\Pi_i(s,t,u)$ function (which only has a cut in s)

Unitarity relations for helicity amplitudes

Helicity amplitudes admit a partial wave expansion

$$H_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s,t,u) = \sum_J D^J(z_s) h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s)$$

where $D^J(z_s)$ is the appropriate Wigner function.

Each partial wave satisfies a simple unitarity relation (for $s > 0$)

$$\mathrm{Im} h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s) = \frac{\sigma_s}{16\pi}\theta(s-4m_\pi^2)\, h_{J,\lambda_1\lambda_2}(s;q_1^2,q_2^2) h^*_{J,\lambda_3\lambda_4}(s;q_3^2,0)$$

where $h_{J,\lambda_1\lambda_2}(s;q_1^2,q_2^2)$ are partial-wave helicity amplitudes of the subprocess $\gamma^*\gamma^* \to \pi\pi$.

Dispersion relations for the $\Pi_i(s,t,u)$

- The $\Pi_i(s,t,u)$ only have a cut in s and for $s \geq 4m_\pi^2$
- Their imaginary part coincides with that of the related helicity amplitude
- The latter can be expanded in partial waves and for each of them unitarity fixes the imaginary part in terms of partial-wave helicity amplitudes of the subprocess $\gamma^*\gamma^* \to \pi\pi$
- a dispersive integral over the right-hand cut of each partial wave would in principle allow me to reconstruct the whole $\Pi_i(s,t,u)$, up to a polynomial

Simplified dispersion relations for the $\Pi_i(s,t,u)$

We will carry out the program outlined in the previous slide with one simplification (but see later!):

for each $\Pi_i(s,t,u)$ we only keep the discontinuity due to the lowest partial wave (*i.e.* S or D)

all $\Pi_i(s,t,u)$ become single-variable functions $\Rightarrow \Pi_i(s)$

This is analogous to what is done for $\pi\pi$ scattering, $\eta \to 3\pi$ and several other processes when the amplitude is expressed as a sum of single-variable functions having only a right-hand cut, and goes under the name of "reconstruction theorem"

Stern, Sazdjian, Fuchs (93)

Fixing subtraction constants: soft-photon zeros

Gauge-invariance implies the presence of so-called soft-photon zeros Low (58), Moussallam (13)

$$H_{\lambda_1\lambda_2,\lambda_3\lambda_4} \stackrel{k\to 0}{\to} \propto (s - q_3^2)$$

and analogously

$$H_{\lambda_1\lambda_2,\lambda_3\lambda_4} \stackrel{q_{1,2}\to 0}{\to} \propto (s - q_{2,1}^2)$$

In a dispersive representation such a property must emerge from the kernels of the dispersive integrals

and constrains the subtraction polynomial

Soft-photon zeros in $\gamma^*\gamma^* \to \pi\pi$

These soft-photon zeros can be studied also in subprocess $\gamma^*\gamma^* \to \pi\pi$ where a dispersive representation for the helicity amplitudes reads

$$h_{J,i}(s) = \frac{1}{\pi} \sum_{J' \text{ even}} \sum_{j=1}^{5} \int_{4m_\pi^2}^{\infty} ds' K_{JJ'}^{ij}(s,s') \,\mathrm{Im} h_{J',j}(s') + \cdots , \quad i,j \in \{\lambda_1\lambda_2\}$$

the ellipsis stands for integrals of crossed-channel partial waves

The diagonal kernel functions

$$K_{00}^{++,++}(t,t') = K_{00}^{00,00}(t,t') = \frac{1}{t'-t} - \frac{t'-q_1^2-q_2^2}{\lambda(t',q_1^2,q_2^2)}$$

$$K_{22}^{++,++}(t,t') = K_{22}^{00,00}(t,t') = \frac{p_t^2 q_t^2}{p_t'^2 q_t'^2}\left(\frac{1}{t'-t} - \frac{t'-q_1^2-q_2^2}{\lambda(t',q_1^2,q_2^2)}\right)$$

display the desired soft-photon behaviour

Soft-photon zeros in $\gamma^*\gamma^* \to \pi\pi$

Soft-photon zeros of the $\gamma^*\gamma^* \to \pi\pi$ sub-amplitudes manifest themselves as a modification of the Cauchy kernel by a factor:

$$K_{12}(s,s') = \frac{f_{12}(s,s')}{s'-s}, \qquad K_{34}(s,s') = \frac{f_{34}(s,s')}{s'-s},$$

for the initial- and final-state photon pair, respectively.

A modified Cauchy kernel that gives the HLbL tensor the proper soft-photon zeros is obtained by factorization

$$K_{12,34}(s,s') = \frac{f_{12}(s,s')f_{34}(s,s')}{s'-s}.$$

Dispersion relations for the $\Pi_i(s)$

Imposing the same form of the soft-photon zeros as in the subamplitudes $\gamma^*\gamma^* \to \pi\pi$ we obtain the following dispersion relations:

$$\Pi_1^s = \bar{h}^0_{++,++}(s) = \frac{s-q_3^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left(\frac{1}{s'-s}-\frac{s'-q_1^2-q_2^2}{\lambda'_{12}}\right)\mathrm{Im}\bar{h}^0_{++,++}(s')$$

$$y\Pi_2^s = \bar{h}^0_{00,++}(s) = \frac{s-q_3^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left(\frac{1}{s'-s}-\frac{s'-q_1^2-q_2^2}{\lambda'_{12}}\right)\mathrm{Im}\bar{h}^0_{00,++}(s')$$

with $y = -\frac{q_1^2 q_2^2}{\xi_1 \xi_2}$ [and similarly for the others]

Dispersion relations for the $\Pi_i(s)$

Soft-photon zeros for the Π_i^s or for the helicity amplitudes?

Remember

$$\bar{H}_{++,++}(s,t,u) = \Pi_1^s + \hat{H}_{++,++}(s,t,u)$$

with

$$\hat{H}_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s,t,u) = \sum_{i=1}^{15} \left(f^i_{\lambda_1\lambda_2,\lambda_3\lambda_4} \Pi_i^t + \tilde{f}^i_{\lambda_1\lambda_2,\lambda_3\lambda_4} \Pi_i^u \right)$$

Dispersion relations for the $\Pi_i(s)$

Soft-photon zeros for the Π_i^s or for the helicity amplitudes?

Remember

$$\bar{H}_{++,++}(s,t,u) = \Pi_1^s + \hat{H}_{++,++}(s,t,u)$$

with

$$\hat{H}_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s,t,u) = \sum_{i=1}^{15}\left(f^i_{\lambda_1\lambda_2,\lambda_3\lambda_4}\Pi_i^t + \tilde{f}^i_{\lambda_1\lambda_2,\lambda_3\lambda_4}\Pi_i^u\right)$$

By sheer kinematics the soft-photon zeros imposed on the Π_i^s imply the correct soft-photon zeros to the full helicity amplitudes

Our dispersive representation of the HLbL tensor

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \left(A_{i,s}^{\mu\nu\lambda\sigma} \Pi_i(s) + A_{i,t}^{\mu\nu\lambda\sigma} \Pi_i(t) + A_{i,u}^{\mu\nu\lambda\sigma} \Pi_i(u) \right)$$

- the $\Pi_i(s)$ are single-variable functions having only a right-hand cut
- for the discontinuity we keep only the lowest partial wave
- the dispersive integral that gives the $\Pi_i(s)$ in terms of its discontinuity has the required soft-photon zeros
- soft-photon zeros constrain the subtraction polynomial to vanish (unless one wanted to subtract more, which is unnecessary)

Contribution of $\bar{\Pi}_{\mu\nu\lambda\sigma}$ to a_μ

$$a_\mu = \lim_{k\to 0} \mathrm{Tr}\left\{ (\not p + m)\Lambda^\rho(p',p)(\not p' + m)\Gamma_\rho(p',p) \right\}$$

$$\Gamma_\rho = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 q_3^2} \frac{\gamma^\mu(\not p' + \not q_1 + m)\gamma^\lambda(\not p - \not q_2 + m)\gamma^\nu}{((p'+q_1)^2 - m^2)((p-q_2)^2 - m^2)} k^\sigma \partial_{k^\rho} \Pi_{\mu\nu\lambda\sigma}$$

A technical caveat: a disadvantage of the basis we chose is that the helicity amplitudes have kinematical singularities – the full HLbL tensor, however, doesn't.

$\Rightarrow$ In order to make sense of the limit $k_\mu \to 0$ for $\bar{\Pi}_{\mu\nu\lambda\sigma}$ we must average over the direction of k_μ first

Contribution of $\bar{\Pi}_{\mu\nu\lambda\sigma}$ to a_μ

$$a_\mu = \frac{1}{16m} \mathrm{Tr}\Big\{ (\not{p} + m) [\gamma^\rho, \gamma^\tau] (\not{p} + m) \tilde{\Gamma}_{\rho\tau} \Big\}$$

$$\tilde{\Gamma}_{\rho\tau} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 s} \frac{\gamma^\mu (\not{p} + \not{q}_1 + m) \gamma^\lambda (\not{p} - \not{q}_2 + m) \gamma^\nu}{\left((p+q_1)^2 - m^2\right)\left((p-q_2)^2 - m^2\right)}$$

$$\times \left[\int \frac{d\Omega(p,k)}{4\pi} \frac{k_\tau k^\sigma}{k^2} \frac{\partial}{\partial k^\rho} \bar{\Pi}_{\mu\nu\lambda\sigma} \right]_{k=0}$$

Contribution of $\bar{\Pi}_{\mu\nu\lambda\sigma}$ to a_μ

$$a_\mu = \frac{1}{16m} \mathrm{Tr}\Big\{ (\not{p} + m) [\gamma^\rho, \gamma^\tau] (\not{p} + m) \tilde{\Gamma}_{\rho\tau} \Big\}$$

$$\tilde{\Gamma}_{\rho\tau} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 s} \frac{\gamma^\mu (\not{p} + \not{q}_1 + m) \gamma^\lambda (\not{p} - \not{q}_2 + m) \gamma^\nu}{\big((p+q_1)^2 - m^2\big)\big((p-q_2)^2 - m^2\big)} \times \left[\int \frac{d\Omega(p,k)}{4\pi} \frac{k_\tau k^\sigma}{k^2} \frac{\partial}{\partial k^\rho} \bar{\Pi}_{\mu\nu\lambda\sigma} \right]_{k=0}$$

- all $A^i_{\mu\nu\rho\sigma}$ tensors scale like $\mathcal{O}(k^0)$
- any term of $\mathcal{O}(k^2)$ in the $\Pi_i(s)$ does not contribute to a_μ
- higher partial waves in $\Pi_i(s)$ are suppressed by angular momentum factors:

$$q_{34}^2 = (s - q_3^2)^2/(4s) = \mathcal{O}(k^2)$$

- $\Rightarrow$ keeping only the lowest partial wave in the discontinuity of the $\Pi_i(s)$ is not an approximation for the calculation of a_μ

Master formula

$$a_\mu^{\pi\pi} = e^6 \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{I^{\pi\pi}}{q_1^2 q_2^2 s \left((p+q_1)^2 - m^2\right)\left((p-q_2)^2 - m^2\right)},$$

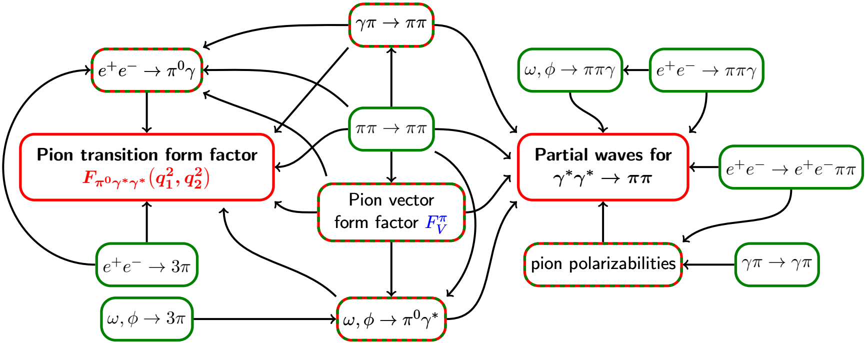
$$I^{\pi\pi} = \sum_{i \in \{1,2,3,6,14\}} \Big(T_{i,s} I_{i,s} + 2 T_{i,u} I_{i,u} \Big) + 2 T_{9,s} I_{9,s} + 2 T_{9,u} I_{9,u} + 2 T_{12,u} I_{12,u}$$

with $I_{i,(s,u)}$ dispersive integrals and $T_{i,(s,u)}$ integration kernels

Conclusions and outlook

- I have presented a dispersive framework for the calculation of the HLbL contribution to a_μ
- which takes into account only single- and double-pion intermediate states
  the extension to other single-particle intermediate states (η, η', etc.) is trivial
- we have derived a master formula which expresses the contribution of 2π intermediate states to a_μ in terms of (integrals over) $\gamma^*\gamma^* \to \pi\pi$ partial waves
- a numerical evaluation of the master formula is in progress
- we believe that this is a step towards a model-independent calculation of the HLbL contribution to a_μ

Hadronic light-by-light: a roadmap



Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among theorists and experimentalists

SM contributions to $(g-2)_\mu$: QED

$a_\mu^{QED} = (1/2)(\alpha/\pi)$ Schwinger 1948

$+ \; 0.765857426\,(16)\,(\alpha/\pi)^2$

Sommerfield; Petermann; Suura&Wichmann '57; Elend '66; MP '04

$+ \; 24.05050988\,(28)\,(\alpha/\pi)^3$

Remiddi, Laporta, Barbieri ... ; Czarnecki, Skrzypek; MP '04;
Friot, Greynat & de Rafael '05, Mohr, Taylor & Newell 2012

$+ \; 130.8796\,(63)\,(\alpha/\pi)^4$

Kinoshita & Lindquist '81, ... , Kinoshita & Nio '04, '05;
Aoyama, Hayakawa, Kinoshita & Nio, 2007, Kinoshita et al. 2012,
Steinhauser et al. 2013 (analytic, in progress).

$+ \; 753.29\,(1.04)\,(\alpha/\pi)^5$ **COMPLETED!**

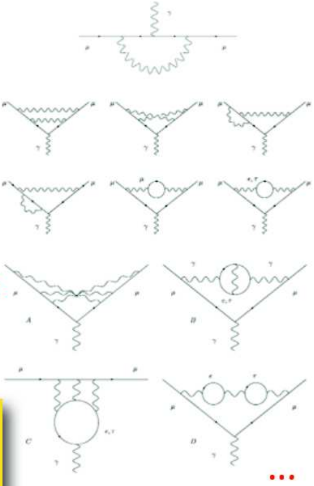
Kinoshita et al. '90, Yelkhovsky, Milstein, Starshenko, Laporta,
Karshenboim,..., Kataev, Kinoshita & Nio '06, Kinoshita et al. 2012

Adding up, we get:

$a_\mu^{QED} = 116584718.951\,(22)(77) \times 10^{-11}$

from coeffs, mainly from 4-loop unc ⟵ ⟶ from $\delta\alpha$(Rb)

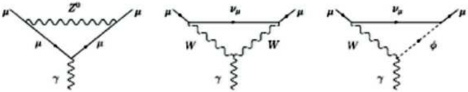
with $\alpha = 1/137.035999049(90)$ [0.66 ppb]



Slide by Massimo Passera

SM contributions to $(g-2)_\mu$: electroweak

- **One-loop term:**



$$a_\mu^{\mathrm{EW}}(\text{1-loop}) = \frac{5G_\mu m_\mu^2}{24\sqrt{2}\pi^2}\left[1+\frac{1}{5}\left(1-4\sin^2\theta_W\right)^2+O\left(\frac{m_\mu^2}{M_{Z,W,H}^2}\right)\right] \approx 195\times10^{-11}$$

1972: Jackiv, Weinberg; Bars, Yoshimura; Altarelli, Cabibbo, Maiani; Bardeen, Gastmans, Lautrup; Fujikawa, Lee, Sanda; Studenikin et al. '80s

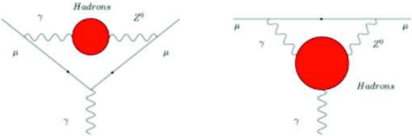
- **One-loop plus higher-order terms:**

$a_\mu^{EW} = 153.6\,(1) \times 10^{-11}$

with M_{Higgs} = 125.6 (1.5) GeV

Hadronic loop uncertainties and 3-loop nonleading logs.

Kukhto et al. '92; Czarnecki, Krause, Marciano '95; Knecht, Peris, Perrottet, de Rafael '02; Czarnecki, Marciano and Vainshtein '02; Degrassi and Giudice '98; Heinemeyer, Stockinger, Weiglein '04; Gribouk and Czarnecki '05; Vainshtein '03; Gnendiger, Stockinger, Stockinger-Kim 2013.



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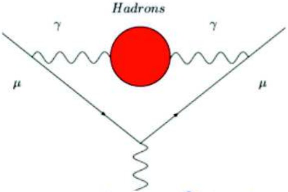
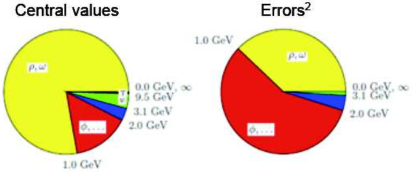
SM contributions to $(g-2)_\mu$: HVP





F. Jegerlehner and A. Nyffeler, Phys. Rept. 477 (2009) 1

$$K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)s/m_\mu^2}$$

$$a_\mu^{\mathrm{HLO}} = \frac{1}{4\pi^3}\int_{4m_\pi^2}^{\infty} ds\, K(s)\, \sigma^{(0)}(s) = \frac{\alpha^2}{3\pi^2}\int_{4m_\pi^2}^{\infty} \frac{ds}{s} K(s)\, R(s)$$

$a_\mu^{\mathrm{HLO}} = 6903\ (53)_{\mathrm{tot}} \times 10^{-11}$ — F. Jegerlehner, A. Nyffeler, Phys. Rept. 477 (2009) 1

$= 6923\ (42)_{\mathrm{tot}} \times 10^{-11}$ — Davier et al, EPJ C71 (2011) 1515 (incl. BaBar & KLOE10 2π)

$= 6949\ (37)_{\mathrm{exp}}\ (21)_{\mathrm{rad}} \times 10^{-11}$ — Hagiwara et al, JPG 38 (2011) 085003

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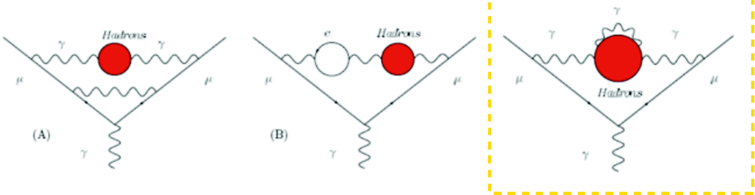
SM contributions to $(g-2)_\mu$: Higher-order HVP

- **HHO: Vacuum Polarization**

Already included in a_μ^{HLO}



O(α^3) contributions of diagrams containing hadronic vacuum polarization insertions:

$$a_\mu^{HHO}(vp) = -98\ (1) \times 10^{-11}$$

Krause '96, Alemany et al. '98, Hagiwara et al. 2011

Only tiny shifts if τ data are used instead of the e^+e^- ones

Davier & Marciano '04.

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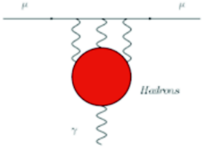
SM contributions to $(g-2)_\mu$: hadronic light-by-light

- **HHO: Light-by-light contribution**



- **Unlike the HLO term, for the hadronic l-b-l term we must rely on theoretical approaches.**
- **This term had a troubled life! Latest values:**

$a_\mu^{HHO}(lbl) = +80\ (40) \times 10^{-11}$	Knecht & Nyffeler '02
$a_\mu^{HHO}(lbl) = +136\ (25) \times 10^{-11}$	Melnikov & Vainshtein '03
$a_\mu^{HHO}(lbl) = +105\ (26) \times 10^{-11}$	Prades, de Rafael, Vainshtein '09
$a_\mu^{HHO}(lbl) = +116\ (39) \times 10^{-11}$	Jegerlehner & Nyffeler '09

Results based also on Hayakawa, Kinoshita '98 & '02; Bijnens, Pallante, Prades '96 & '02

- **"Bound" $a_\mu^{HHO}(lbl) <\sim 160 \times 10^{-11}$** Erler, Sanchez '06, Pivovarov '02; also Boughezal, Melnikov '11
- **Lattice? Very hard... in progress.** M. Golterman @ PhiPsi 2013; T. Blum @ Lattice 2012
- **Pion exch. in holographic QCD agrees.** D.K.Hong, D.Kim '09; Cappiello, Catà, D'Ambrosio '11
- **"By far not complete" calculation: 188×10^{-11}** Fischer et al, PRD87(2013)034013
- **Had lbl is likely to become the ultimate limitation of the SM prediction.**

Slide by Massimo Passera

SM contributions to $(g-2)_\mu$:

$a_\mu^{EXP} = 116592089\ (63) \times 10^{-11}$

E821 – Final Report: PRD73 (2006) 072 with latest value of $\lambda = \mu_\mu/\mu_p$ from CODATA'06

$a_\mu^{SM} \times 10^{11}$	$\Delta a_\mu = a_\mu^{EXP} - a_\mu^{SM}$	σ
116 591 793 (66)	$296\ (91) \times 10^{-11}$	3.2 [1]
116 591 813 (57)	$276\ (85) \times 10^{-11}$	3.2 [2]
116 591 839 (58)	$250\ (86) \times 10^{-11}$	2.9 [3]

with the "conservative" a_μ^{HHO}(lbl) = 116 (39) x 10^{-11} and the LO hadronic from:

[1] Jegerlehner & Nyffeler, Phys. Rept. 477 (2009) 1
[2] Davier et al, EPJ C71 (2011) 1515 (includes BaBar & KLOE10 2π)
[3] Hagiwara et al, JPG38 (2011) 085003 (includes BaBar & KLOE10 2π)

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