Conformal Tensors in Lovelock Gravity

Lovelock gravity shares important features with Einstein gravity…

But exactly which features?

Do we know them all?

The Riemann-Lovelock Curvature Tensor
David Kastor (Massachusetts U., Amherst). Feb 2012. 12 pp.
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Conformal Tensors via Lovelock Gravity
David Kastor (Massachusetts U., Amherst). Jun 19, 2013. 17 pp.
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Formulate some basic questions…

a) Higher Curvature Bianchi Identities
b) Analogues of 3D GR

1. Intro to Lovelock
2. Riemann-Lovelock Tensor & "Lovelock Flatness"
3. Weyl-Lovelock et. al.
4. Further (interesting?) questions

Answers supplied by new higher curvature constructs

What about "conformal Lovelock flatness"?

In abundance…

1) Introduction to Lovelock

Coupling constants

Higher curvature analogues of scalar curvature

$$\boxed{S = \int d^D x \sqrt{-g} \sum_{k=0}^{k_D} a_k \mathcal{R}^{(k)}}$$

$$\mathcal{R}^{(k)} = \delta^{a_1 \dots a_{2k}}_{b_1 \dots b_{2k}} \, R_{a_1 a_2}{}^{b_1 b_2} \, \dots \, R_{a_{2k-1} a_{2k}}{}^{b_{2k-1} b_{2k}}$$

$$\delta^{b_1 \dots b_n}_{a_1 \dots a_n} = \delta^{b_1}_{[a_1} \cdots \delta^{b_n}_{a_n]}$$

$$\mathcal{R}^{(0)} = 1$$ Cosmological constant term

$$\mathcal{R}^{(1)} = R$$ Einstein-Hilbert term

$$\mathcal{R}^{(2)} = \frac{1}{6} \left(R_{ae}{}^{cd} R_{cd}{}^{be} - 2 R_{ad}{}^{bc} R_c{}^d - 2 R_c{}^b R_a{}^c + R_a{}^b R \right)$$ Gauss-Bonnet term

$$\mathcal{R}^{(k)}$$

Euler density in D=2k dimensions

- vanishes for D<2k
- variation vanishes in D=2k

$$k_D = \left[\frac{D-1}{2} \right]$$

The nice thing about Lovelock…

Equations of motion depend only on Riemann tensor and not its derivatives

- no 4th derivatives of metric
- same initial data as GR
- no ghosts

$$\boxed{S = \int d^D x \sqrt{-g} \sum_{k=0}^{k_D} a_k \mathcal{R}^{(k)}}$$

$$\sum_{k=0}^{k_D} a_k \mathcal{G}^{(k)a}{}_b = 0$$

Higher curvature analogues of Einstein tensor

$$\mathcal{G}_a^{(k)b} = \frac{(2k+1)\alpha_k}{2} \delta^{bc_1\ldots c_{2k}}_{ad_1\ldots d_{2k}} R_{c_1c_2}{}^{d_1d_2} \ldots R_{c_{2k-1}c_{2k}}{}^{d_{2k-1}d_{2k}}$$

$$\nabla_a \mathcal{G}^{(k)a}{}_b = 0$$

Covariantly conserved

$\mathcal{G}^{(1)a}{}_b$ Einstein tensor

$\mathcal{G}^{(k)a}{}_b$ Vanishes for D < 2k+1

Questions…

A) $\nabla_a \mathcal{G}^{(k)a}{}_b = 0$ Covariantly conserved

k=1 Follows from twice contracted Bianchi identity

$$\nabla_{[a} R_{bc]}{}^{de} = 0 \quad \Rightarrow \quad \begin{aligned} 0 &= \nabla_{[a} R_{bc]}{}^{bc} \\ &= \frac{1}{3}(\nabla_a R - 2\nabla_b R^b{}_a) \\ &= -\frac{2}{3}\nabla_b G^b{}_a \end{aligned}$$

Is there an analogue of the uncontracted Bianchi identity for k>1?

Is there a higher curvature Lovelock analogue of the Riemann tensor in this sense?

Questions…

B) Vacuum GR in D=3

$$G_{ab} = 0 \Rightarrow R_{ab}{}^{cd} = 0$$

All solutions to Einstein's equation are flat

Both Riemann and Ricci tensors have 6 independent components

3 x 3 symmetric tensors

Or simple Lovelock-type construction …

$$\delta^{cdef}_{abgh} R_{ef}{}^{gh} = \frac{1}{6}\left(R_{ab}{}^{cd} - 4\delta^{[c}_{[a} R_{b]}{}^{d]} + \delta^{cd}_{ab} R\right)$$

Relation is true in all dimensions

LHS vanishes in D=3, determining Riemann tensor in terms of its contractions

Questions…

B) Vacuum GR in D=3

$$G_{ab} = 0 \Rightarrow R_{ab}{}^{cd} = 0$$

All solutions to Einstein's equation are flat

Is there an analogue of this for k>1?

$\mathcal{R}^{(k)}$ Euler density in D=2k dimensions $\Rightarrow$ Look at "pure" k^{th} order Lovelock gravity in D=2k+1

$$\boxed{S_{pure} = \int d^D x \sqrt{-g}\mathcal{R}^{(k)}}$$

Only k^{th} order Lovelock term in action

Trivial in D=2k, like GR in D=2

D=2k+1 $\Rightarrow$ This is highest order Lovelock term available

Expect all solutions to Lovelock will asymptote to solutions of pure k^{th} order theory in high curvature regime

Questions…

B) Vacuum GR in D=3

$$G_{ab} = 0 \Rightarrow R_{ab}{}^{cd} = 0$$

All solutions to Einstein's equation are flat

Is there an analogue of this for k>1?

$\boxed{\mathcal{R}^{(k)} \text{ Euler density in D=2k dimensions}}$ $\Rightarrow$ Look at "pure" k^{th} order Lovelock gravity in D=2k+1

$$\boxed{S_{pure} = \int d^D x \sqrt{-g}\mathcal{R}^{(k)}}$$

Only k^{th} order Lovelock term in action

Is there a higher curvature Lovelock flatness condition, such that all solutions to pure k^{th} order Lovelock in D=2k+1 are k^{th} order Lovelock flat?

Like question 1, this calls for a higher curvature analogue of the Riemann tensor

2) Riemann-Lovelock tensors & "Lovelock flatness"

Call this Riemann$^{(k)}$ tensor

$$\mathcal{R}^{(k)}{}_{a_1 b_1 \ldots a_k b_k}{}^{c_1 d_1 \ldots c_k d_k} \equiv R_{[a_1 b_1}{}^{[c_1 d_1} R_{a_2 b_2}{}^{c_2 d_2} \cdots R_{a_k b_k]}{}^{c_k d_k]}$$

Tensor of type (2k,2k), vanishes for D<2k and satisfies…

Like all 1D spaces are k=1 Lovelock flat

$$\mathcal{R}^{(k)}{}_{a_1 \ldots a_{2k} b_1 \ldots b_{2k}} = \mathcal{R}^{(k)}{}_{[a_1 \ldots a_{2k}] b_1 \ldots b_{2k}} = \mathcal{R}^{(k)}{}_{a_1 \ldots a_{2k} [b_1 \ldots b_{2k}]} = \mathcal{R}^{(k)}{}_{b_1 \ldots b_{2k} a_1 \ldots a_{2k}}$$

Symmetries

$$\mathcal{R}^{(k)}{}_{[a_1 \ldots a_{2k} b_1]}{}^{b_2 \ldots b_{2k}} = 0$$

$$\nabla_{[c} \mathcal{R}^{(k)}{}_{a_1 \ldots a_{2k}]}{}^{b_1 \ldots b_{2k}} = 0$$

Bianchi identities

Analogous to familiar properties of Riemann tensor

k^{th} order Lovelock flatness or Riemann$^{(k)}$ flat

$$\mathcal{R}^{(k)}{}_{a_1 b_1 \ldots a_k b_k}{}^{c_1 d_1 \ldots c_k d_k} = 0$$

Taking traces…

$$\mathcal{R}^{(k)} = \mathcal{R}^{(k)}_{a_1 \ldots a_{2k}}{}^{a_1 \ldots a_{2k}}$$

Tracing over all pairs of indices gives back scalar Lovelock interaction terms

$$\mathcal{R}_a^{(k)\,b} = \mathcal{R}^{(k)}_{a c_1 \ldots c_{2k-1}}{}^{b c_1 \ldots c_{2k-1}}$$

Ricci$^{(k)}$ tensor is an analogue of Ricci tensor

$$\mathcal{G}_a^{(k)\,b} = k\mathcal{R}_a^{(k)\,b} - (1/2)\delta_b^a \mathcal{R}^{(k)}$$

Einstein$^{(k)}$ tensor appears in Lovelock equation of motion

$$\begin{aligned} 0 =& \nabla_{[a} \mathcal{R}^{(k)}_{b_1 \ldots b_{2k}]}{}^{b_1 \ldots b_{2k}} \\ =& \frac{1}{2k+1}\left(\nabla_a \mathcal{R}^{(k)} - 2k \nabla_b \mathcal{R}^{(k)b}{}_a\right) \\ =& -\frac{2}{2k+1}\nabla_b \mathcal{G}^{(k)b}{}_a \end{aligned}$$

Fully contracted Bianchi identity yields vanishing divergrance for Einstein$^{(k)}$ tensors

Answers 1st question

Demonstrates some relevance for Riemann-Lovelock tensors

Pure kth order Lovelock in D=2k+1

Analogue of
vacuum GR in D=3

$$S_{pure} = \int d^{2k+1}x\sqrt{-g}\mathcal{R}^{(k)}$$

$$\mathcal{G}_a^{(k)b} = k\mathcal{R}_a^{(k)b} - (1/2)\delta_b^a\mathcal{R}^{(k)} = 0$$

Yes

2nd Question ➡ Are all solutions kth order Lovelock flat?

$$\mathcal{R}^{(k)}_{a_1b_1\ldots a_kb_k}{}^{c_1d_1\ldots c_kd_k} \equiv R_{[a_1b_1}{}^{[c_1d_1}R_{a_2b_2}{}^{c_2d_2}\cdots R_{a_kb_k]}{}^{c_kd_k]}$$

D=2k+1 ➡ Same number of independent components
as symmetric (2k+1)x(2k+1) tensor

Can show $\mathcal{R}_a^{(k)b} = 0$ ➡ $\mathcal{R}^{(k)}_{a_1b_1\ldots a_kb_k}{}^{c_1d_1\ldots c_kd_k} = 0$

Are there interesting spacetimes that are higher order Lovelock flat, but not Riemann flat?

Large set of examples…

Riemann$^{(k)}$ tensor vanishes for any spacetime of dimension D < 2k

Can build higher dimensional Riemann$^{(k)}$ flat spacetimes by adding flat directions

Interesting example in D=2k+1…

Static, spherically symmetric solutions of pure kth order Lovelock are missing solid angle spacetimes

$$ds^2_{2k+1} = -dt^2 + dr^2 + \alpha^2 r^2 d\Omega^2_{2k-1}$$

Static, spherically symmetric solutions of pure kth order Lovelock are missing solid angle spacetimes

$$ds^2_{2k+1} = -dt^2 + dr^2 + \alpha^2 r^2 d\Omega^2_{2k-1}$$

$$R_{\mu\nu}{}^{\rho\sigma} = \frac{2}{\alpha^2\, r^2}(1-\alpha^2)\delta^{\rho\sigma}_{\mu\nu} \qquad \mu,\nu = 1,\ldots,2k-1$$

Angular coordinates on sphere

Only nonzero curvature components

Curved for $\alpha \neq 1$

Riemann$^{(k)}$ tensor

$$\mathcal{R}^{(k)}_{a_1 b_1 \ldots a_k b_k}{}^{c_1 d_1 \ldots c_k d_k} \equiv R_{[a_1 b_1}{}^{[c_1 d_1} R_{a_2 b_2}{}^{c_2 d_2} \cdots R_{a_k b_k]}{}^{c_k d_k]} = 0$$

Involves anti-symmetrization over 2k indices, but only 2k-1 are available…

Static, spherically symmetric solutions of pure kth order Lovelock are missing solid angle spacetimes

$$ds^2_{2k+1} = -dt^2 + dr^2 + \alpha^2 r^2 d\Omega^2_{2k-1}$$

$$R_{\mu\nu}{}^{\rho\sigma} = \frac{2}{\alpha^2 r^2}(1-\alpha^2)\delta^{\rho\sigma}_{\mu\nu} \qquad \mu,\nu = 1,\ldots,2k-1$$

Angular coordinates on sphere

Only nonzero curvature components

Curved for $\alpha \neq 1$

GR in D=3 → k=1

$$ds^2_3 = -dt^2 + dr^2 + \alpha^2 d\phi^2$$

Missing angle

Flat

Global flat space with identifications

General case

k>1 →

Missing *solid* angle

Riemann$^{(k)}$ Flat

~~Global flat space with identifications~~

Further question…

Can we classify all Riemann$^{(k)}$ flat spacetimes?

Conformal tensors in Lovelock…

Riemann$^{(k)}$ flatness $\xrightarrow{\text{Next step}}$ Conformal$^{(k)}$ flatness

A spacetime is Conformal$^{(k)}$ flat if it is related to to a Riemann$^{(k)}$ flat spacetime via a conformal transformation

$D \geq 4$ Conformal flatness $\longleftrightarrow$ Weyl tensor vanishes

Trace free part of Riemann tensor

$\Rightarrow$ Consider trace free part of Riemann$^{(k)}$ tensors $\Rightarrow$ Weyl$^{(k)}$ tensors

Do Weyl$^{(k)}$ tensors determine Conformal$^{(k)}$ flatness?

First recall some other constructs…

$$W_{ab}{}^{cd} = R_{ab}{}^{cd} - 4\delta^{[c}_{[a} S_{b]}{}^{d]}$$

Weyl tensor

$$S_a{}^b = \frac{1}{D-2}\left(R_a{}^b - \frac{1}{2(D-1)}\delta^b_a R\right)$$

Schouten tensor

$$C_{ab}{}^c = 2\nabla_{[a} S_{b]}{}^c$$

$$C_{ab}{}^b = 0$$

Cotton tensor

Conformal transformations $\tilde{g}_{ab} = e^{2f} g_{ab}$

$$\tilde{W}_{ab}{}^{cd} = e^{-2f} W_{ab}{}^{cd}$$

$$\tilde{C}_{ab}{}^c = e^{-2f}\left(C_{ab}{}^c - W_{ab}{}^{cd}\nabla_d f\right)$$

Conformal transformations $\tilde{g}_{ab} = e^{2f} g_{ab}$

$$\tilde{W}_{ab}{}^{cd} = e^{-2f} W_{ab}{}^{cd}$$

$$\tilde{C}_{ab}{}^{c} = e^{-2f} \left(C_{ab}{}^{c} - W_{ab}{}^{cd} \nabla_d f \right)$$

Vanishes in D=3

D=3 ➡ $W_{ab}{}^{cd} = 0$

$$\delta^{cdef}_{abgh} W_{ef}{}^{gh} = (1/6) W_{ab}{}^{cd}$$

Cotton tensor is conformally invariant

Conformal flatness condition in D=3 $C_{ab}{}^{c} = 0$

D=2 ➡ Weyl tensor not defined

All metrics are locally conformally flat

Conformal tensors in Lovelock…

Define Weyl$^{(k)}$ tensor as traceless part of Riemann$^{(k)}$ tensor

$$\mathcal{W}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k}} = \mathcal{R}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k}} + \sum_{p=1}^{2k} \alpha_p\, \delta^{[b_1\ldots b_p}_{[a_1\ldots a_p} \mathcal{R}^{(k)}_{a_{p+1}\ldots a_{2k}]}{}^{b_{p+1}\ldots b_{2k}]} \ .$$

$$\alpha_p = \left(\frac{(2k)!}{(2k-p)!}\right)^2 \frac{(-1)^p (D-(4k-1))!}{p!(D-(4k-p-1))!}$$

Can show

Riemann$^{(k)}$ tensor determined by its traces for D<4k

Expect Weyl$^{(k)}$ tensor is nontrivial only for $D \geq 4k$

$D < 4k - 1$ Weyl$^{(k)}$ tensor undefined because of divergent coefficients

Like Weyl tensor in D=1,2

$D = 4k - 1$ Weyl$^{(k)}$ tensor defined, but vanishes identically

Like Weyl tensor in D=3

Schouten$^{(k)}$ and Cotton$^{(k)}$ tensors

$$\mathcal{W}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k}} = \mathcal{R}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k}} - (2k)^2\delta^{[b_1}_{[a_1}\mathcal{S}^{(k)}_{a_2\ldots a_{2k}]}{}^{b_2\ldots b_{2k}]}$$

$$\mathcal{C}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k-1}} = 2k\,\nabla_{[a_1}\mathcal{S}^{(k)}_{a_2\ldots a_{2k}]}{}^{b_1\ldots b_{2k-1}}$$

All in parallel with k=1 case….

$$\mathcal{C}^{(k)}_{a_1\ldots a_{2k-1}c}{}^{b_1\ldots b_{2k-2}c} = 0$$

Traceless

$$\nabla_c\mathcal{W}^{(k)}_{a_1\ldots a_{2k}}{}^{cb_1\ldots b_{2k-1}} = (D-(4k-1))\,\mathcal{C}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k-1}}$$

Conformal transformations $\tilde{g}_{ab} = e^{2f}g_{ab}$

$$\tilde{\mathcal{W}}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k}} = e^{-2kf}\,\mathcal{W}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k}}$$

$$\tilde{\mathcal{C}}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k-1}} = e^{-2kf}\left(\mathcal{C}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k-1}} - \mathcal{W}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k-1}c}\nabla_c f\right)$$

D=4k-1 $\longrightarrow$ Weyl$^{(k)}$ tensor vanishes $\longrightarrow$ Cotton$^{(k)}$ is conformally invariant

Math. Ann. 199, 175—204 (1972)

On the Bianchi Identities

Ravindra S. Kulkarni

No connection to Lovelock, but roughly the same time period

- Demonstrates properties of Riemann$^{(k)}$ tensors
- Defines Weyl$^{(k)}$ tensors and shows conformal invariance

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The Einstein Tensor and Its Generalizations*

David Lovelock
Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada

(Received 27 August 1970)

Conformal transformation of Weyl tensor…

Let $A_{ab}{}^{cd}$ satisfy $A_{abcd} = A_{[ab]cd} = A_{ab[cd]} = A_{cdab}$

Traces $A_a{}^c = A_{ab}{}^{cb}$ $A = A_a{}^a$

Trace free part

$$A^{(t)\,cd}_{ab} = A_{ab}{}^{cd} - \frac{4}{D-2}\delta^{[c}_{[a}A_{b]}{}^{d]} + \frac{2}{(D-1)(D-2)}\delta^{cd}_{ab}A$$

Let $\tilde{A}_{ab}{}^{cd} = A_{ab}{}^{cd} + \delta^{[c}_{[a}\Lambda_{b]}{}^{d]}$ with $\Lambda_{ab} = \Lambda_{ba}$

Can show that….

$$\tilde{A}^{(t)\,cd}_{ab} = A^{(t)\,cd}_{ab}$$

Analogous construction works for all k

Conformal transformation $\tilde{R}_{ab}{}^{cd} = e^{-2f}\left(R_{ab}{}^{cd} + \delta^{[c}_{[a}\Lambda_{b]}{}^{d]}\right)$

$$\Lambda_a{}^b = 4\nabla_a\nabla^b f + 4(\nabla_a f)\nabla^b f - 2\delta^b_a(\nabla_c f)\nabla^c f$$

Result

Conformal$^{(k)}$ flatness conjectures

		k=1 result
$D < 2k$	Riemann$^{(k)}$ tensor vanishes All spacetimes conformal$^{(k)}$ flat	No curvature in D=1
$D = 2k$	Riemann$^{(k)}$ tensor has a single component All spacetimes (locally) conformal$^{(k)}$ flat?	All D=2 spacetimes are (locally) conformally flat
$2k < D < 4k - 1$	Weyl$^{(k)}$ & Cotton$^{(k)}$ tensors not defined All spacetimes (locally) conformal$^{(k)}$ flat??	$2 < D < 3$ No k=1 analogue
$D = 4k - 1$	Conformal$^{(k)}$ flat if Cotton$^{(k)}$ tensor vanishes?	D=3 spacetime is conformally flat if Cotton tensor vanishes
$D \geq 4k$	Conformal$^{(k)}$ flat if Weyl$^{(k)}$ tensor vanishes?	$D \geq 4$ Weyl tensor vanishing implies conformal flatness

New gravity models?

Recall that low dimensional gravity models make use of conformal tensors…

D=3 Topologically massive gravity (Deser, Jackiw & Templeton – 1982)

Cotton tensor appears in equation of motion

D=3 New massive gravity (Bergshoeff, Hohm & Townsend – 2009)

Schouten tensor is ingredient in action

Perhaps conformal$^{(k)}$ tensors can be useful in model building associated with Lovelock theories in low(ish) dimensions…

Simple example $\longrightarrow$ Conformal$^{(k)}$ gravity in D=4k

Recall…

Conformal gravity in D=4 $$S = \int d^4x\sqrt{-g}\, W_{ab}{}^{cd}W_{cd}{}^{ab}$$

Equation of motion $B_{ab} = 0$

Bach tensor $$B_a{}^b = (\nabla^d\nabla_c + \frac{1}{2}R_c{}^d)W_{ad}{}^{bc}$$

Symmetric, traceless

$$\tilde{B}_a{}^b = e^{-4f}B_a{}^b$$

Equations of motion are conformally invariant

All Einstein metrics have vanishing Bach tensor

All conformally Einstein spacetimes are solutions to conformal gravity

Simple example → Conformal$^{(k)}$ gravity in D=4k

$$S = \int d^{4k}x\sqrt{-g}\; \mathcal{W}^{(k)}_{a_1\dots a_{2k}}{}^{b_1\dots b_{2k}}\, \mathcal{W}^{(k)}_{b_1\dots b_{2k}}{}^{a_1\dots a_{2k}}.$$

Equation of motion

Compare with…

$$B_a{}^b = (\nabla^d\nabla_c + \frac{1}{2}R_c{}^d)W_{ad}{}^{bc}$$

Bach tensor

$$\mathcal{B}^{(k)\,b}_{a} = \left(\mathcal{R}^{(k-1)}_{c_1\dots c_{2k-2}}{}^{d_1\dots d_{2k-2}}\nabla^{d_{2k-1}}\nabla_{c_{2k-1}} + \frac{k}{2}\mathcal{R}^{(k)}_{c_1\dots c_{2k-1}}{}^{d_1\dots d_{2k-1}}\right)\mathcal{W}^{(k)}_{ad_1\dots d_{2k-1}}{}^{bc_1\dots c_{2k-1}}$$

$$\mathcal{B}^{(k)}_{(ab)} = 0$$

Expect anti-symmetric part of Bach tensor vanishes, but not straightforward to show…

As it does for k=1

Also expect Bach$^{(k)}$ tensor is a conformal invariant, because of conformal invariance of action

Simple example $\longrightarrow$ Conformal$^{(k)}$ gravity in D=4k

$$S = \int d^{4k}x\sqrt{-g}\; \mathcal{W}^{(k)}_{a_1\ldots a_{2k}}{}^{b_1\ldots b_{2k}}\, \mathcal{W}^{(k)}_{b_1\ldots b_{2k}}{}^{a_1\ldots a_{2k}}.$$

$$\mathcal{B}^{(k)\,b}_{a} = \left(\mathcal{R}^{(k-1)}_{c_1\ldots c_{2k-2}}{}^{d_1\ldots d_{2k-2}}\nabla^{d_{2k-1}}\nabla_{c_{2k-1}} + \frac{k}{2}\mathcal{R}^{(k)}_{c_1\ldots c_{2k-1}}{}^{d_1\ldots d_{2k-1}}\right)\mathcal{W}^{(k)}_{ad_1\ldots d_{2k-1}}{}^{bc_1\ldots c_{2k-1}}$$

$$\mathcal{B}^{(k)}_{(ab)} = 0$$

Solved by Einstein$^{(k)}$ spaces…

$$\mathcal{R}^{(k)}_{a_1\ldots a_{2k-1}}{}^{b_1\ldots b_{2k-1}} = \alpha\,\delta^{b_1\ldots b_{2k-1}}_{a_1\ldots a_{2k-1}}$$

Conclusions…

Riemann$^{(k)}$ tensor looks interesting.

Lots of related questions…