Resurgence out of the (literal) box

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Resurgence for QFT

Belief: QFT observables are transseries in the couplings

$$\mathcal{O}(\lambda) \simeq \sum_n p_n \lambda^n + \sum_c e^{-\frac{S_c}{\lambda}} \sum_k p_{k,c} \lambda^k + \cdots$$

Generically, all series are separately divergent and ambiguous, but $\mathcal{O}(\lambda)$ is well-defined due to devious conspiracies between terms

Why believe this specifically in full QFT?

Very hard to explore high loop orders!

Resurgence for 0d QFT

First explicit check: 0-dimensional "QFT"

$$Z(\lambda) = \frac{1}{\sqrt{2\pi\lambda}} \int_{-\pi/2}^{\pi/2} dx\, e^{-\frac{1}{2\lambda}\sin^2(x)}$$

Can be done very explicitly.

$$Z_{\rm pert} = \sum_n p_n \lambda^n, \;\; p_n \sim \frac{(n-1)!}{\pi(1/2)^n}\left(1 + \frac{(-1/2)(1/2)}{(n-1)} + \frac{(9/8)(1/2)^2}{(n-1)(n-2)} \cdots\right)$$

$$Z_{\rm non-pert}(\lambda) = \mp i e^{-\frac{(1/2)}{\lambda}} \left[1 + (-\frac{1}{2})\lambda + \frac{9}{8}\lambda^2 + (-\frac{75}{16})\lambda^3 + \cdots\right]$$

Resurgence idea works!

Resurgence for QM

Second explicit check: 1-dimensional "QFT" - quantum mechanics!

Detailed explorations focused on QM with smooth potentials V(x)

$$\left[-\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x)$$

Dunne + Unsal 2013:

perturbation theory + finite # of conditions on $\boldsymbol{\psi}$(x) = everything.

$$[0]=\sum_n c_n\lambda^n,\ c_n\sim\frac{2}{\pi}\frac{n!}{(2S_I)^n}\left[a_0-a_1\left(\frac{2S_I}{n}\right)-a_2\left(\frac{2S_I}{n}\right)^2+\cdots\right]$$

$$\mathrm{Im}[\mathcal{I}\bar{\mathcal{I}}]\sim\pm\frac{4S_I}{\lambda}e^{-2S_I/\lambda}\left(a_0-a_1\lambda-a_2\lambda^2+\cdots\right)$$

Resurgence idea works!

Resurgence for QM

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$$\left[-\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x)$$

Resurgence idea works!

Dunne + Unsal 2013:

perturbation theory + finite # of conditions on ψ(x) = everything.

Basar + Dunne 2015

Relation of resurgence to
elliptic curve associated to V(x)

Gives some explanation of \`why’ it works; similar story can be told in 0d.

Resurgence for QFT?

Witten 2009; Dunne, Unsal, AC, Dorigoni, Basar, … 2013-now

Why should the d = 1 results generalize to d > 1?

Elliptic curve picture seems closely tied to QM, generalization unclear.

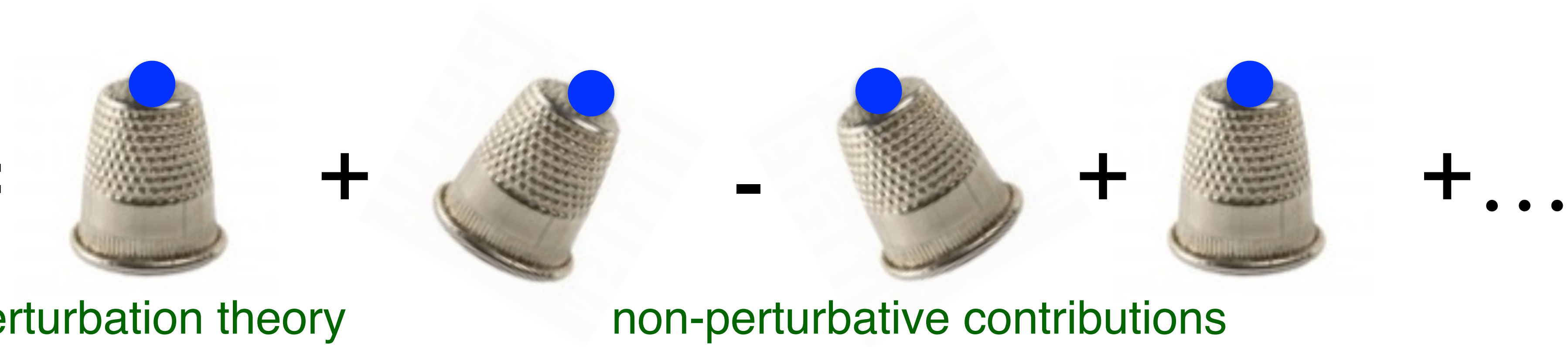
Path integral perspective?

$$Z(\lambda) = \sum_k C_k Z_{J_k}(\lambda)$$

“Lefshetz thimble” integration cycles

One `thimble’ per critical point of classical action, defined by steepest descent.

$Z(\lambda)$ =



{set of thimbles} = complete basis for convergent path integrals

Resurgence relations = jumps in C_k as arg[λ] varies.

Resurgence for QFT?

Thimble perspective might sound taylor-made for generalization to QFT…

… but this isn't obvious!

Witten proved thimble decomposition works in $d = 1 > 0$

No proof that set of critical-point cycles is a basis away from $d = 1$!

Several possibly-related issues.
What counts as a critical point? How to perform decomposition? …

Even in $d = 1$ discontinuous saddle-point-field configurations must be taken into account!

Behtash, Dunne, Schafer, Sulejmanpasic, Unsal, 2015

Construction in $d > 1$ may be sensitive to regularization of integral.

Shouldn't be too shocking: regularization always important in $d > 1$!

Resurgence in QFT

Third explicit check: 1+1D asymptotically-free QFTs

CP^{N-1} , principal chiral, O(N), and Grassmannian non-linear sigma models

To the extent it's been checked, resurgence works!

linear combinations of Dunne, Unsal, AC, Dorigoni
2012-2015

Why the weasel words?

In d > 1 QFT, very difficult to precisely characterize large-order behavior

Strong coupling in IR in asymptotically-free theories

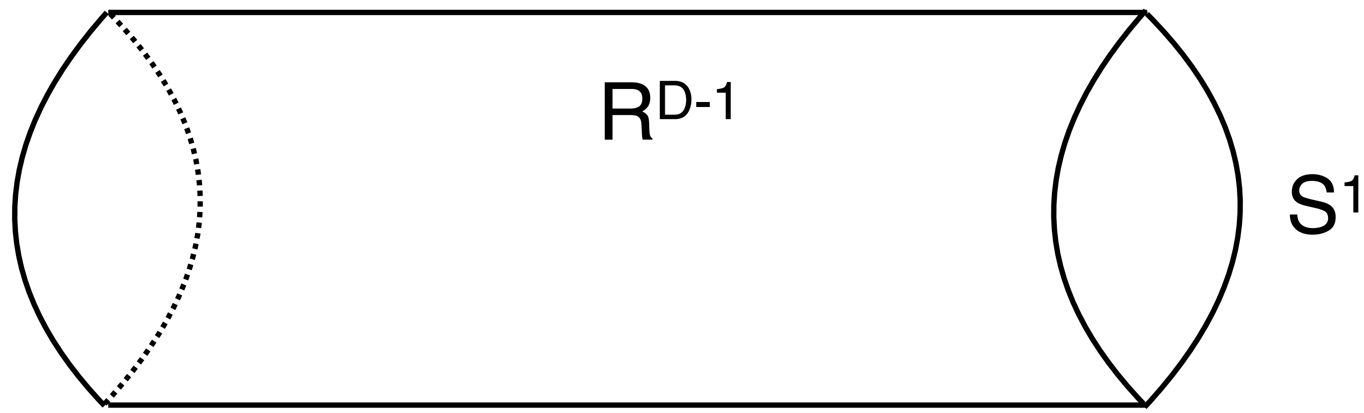
$$\Lambda \approx \mu\, e^{-c/\lambda}\,,\ \lambda = g^2 N$$

All work so far used idea of adiabatic
compactification from R^2 to $R \times S^1$

Tiny boxes as tools

Compactify asymptotically-free QFT from R^D to $R^{D-1} x S^1$

Idea: when S^1 size $L << \Lambda^{-1}$, theory becomes ≈ weakly-coupled



Simplest circle is a thermal one. Trouble: physics at small-L and large-L can look totally different

Examples:

Large N phase transitions as a function of L

Dependence of gap Δ on 2D strong scale Λ is power law at large L, only logarithmic at small L.

Adiabatic small circle limit

For a smooth $L << \Lambda^{-1}$ limit, use special non-thermal boundary conditions.

Idea is actually quite general, very closely related to constructions in 4D gauge theory

Unsal and collaborators, 2012-onward

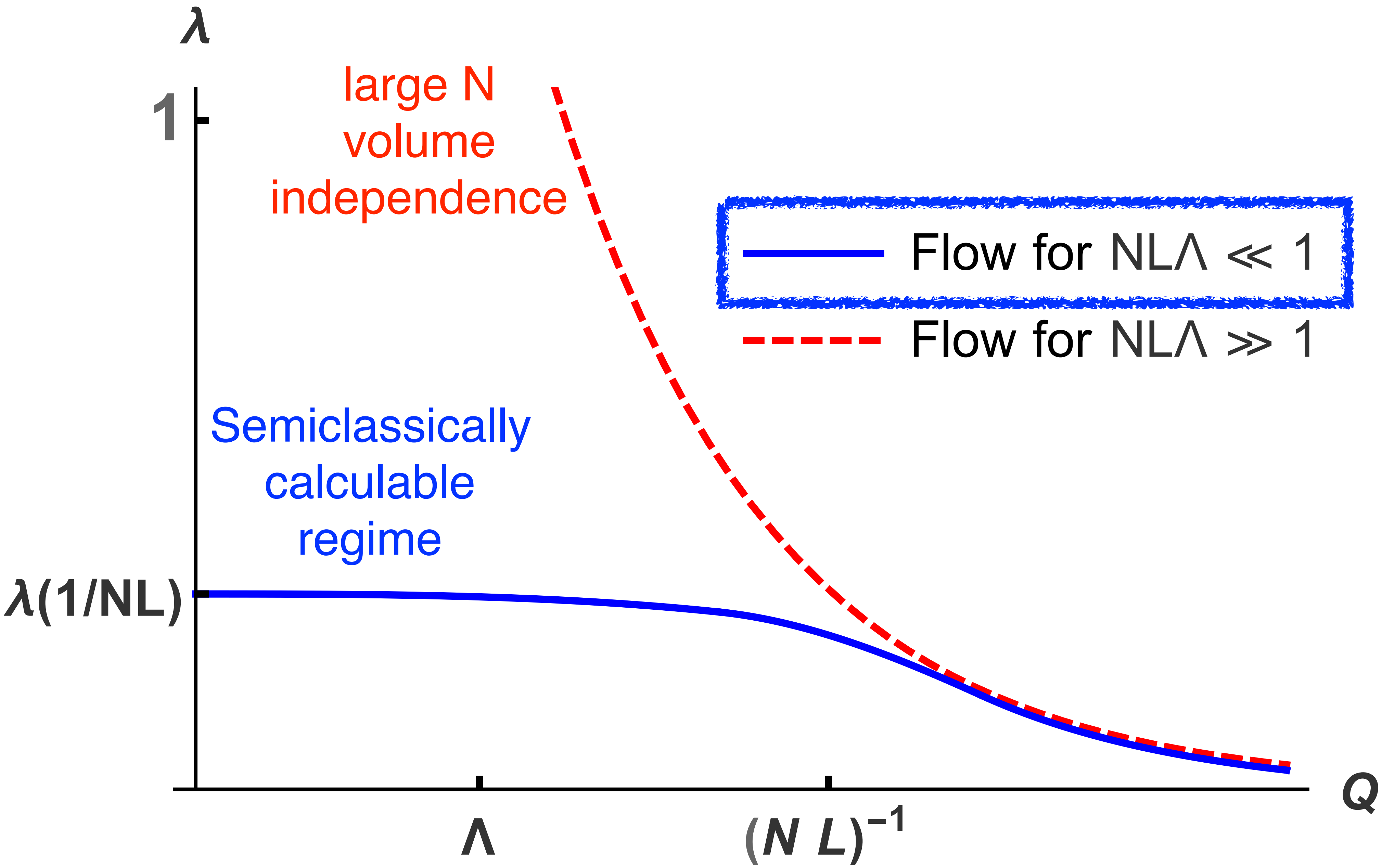
4D gauge theory: adiabatic small-L limit obtained with Z_N-invariant S^1 holonomy for the dynamical gauge field

2D sigma models: adiabatic small-L limit obtained with Z_N-invariant S^1 holonomy for the background `flavor' gauge field

With such compactifications, effective KK scale is 1/(NL), not 1/L.

Large N and small L limits do not commute - tied to large N volume independence!

Coupling flow with adiabatic compactification



NLΛ >> 1 regime is strongly coupled

The NLΛ << 1 regime gives a weakly-coupled theory

Physics is very rich - mass gap, renormalons present at small N L!

Resurgence in a box

In perturbation theory 2D sigma models like O(N), CP^{N-1}, etc are gapless.

What about non-perturbatively, in the small NLΛ limit?

Need to know about non-perturbative saddle points!

The Z_N-invariant holonomies make instantons fractionalize into ~ N constituent `fractons' (or `monopole-instantons', etc.)

Without instantons, what fractionalizes are `unitons' - finite-action, non-BPS saddle-point solutions.

Dabrowski, Dunne; AC, Dorigoni, Dunne, Unsal

Very common in 2D: relevant homotopy group is π_2.

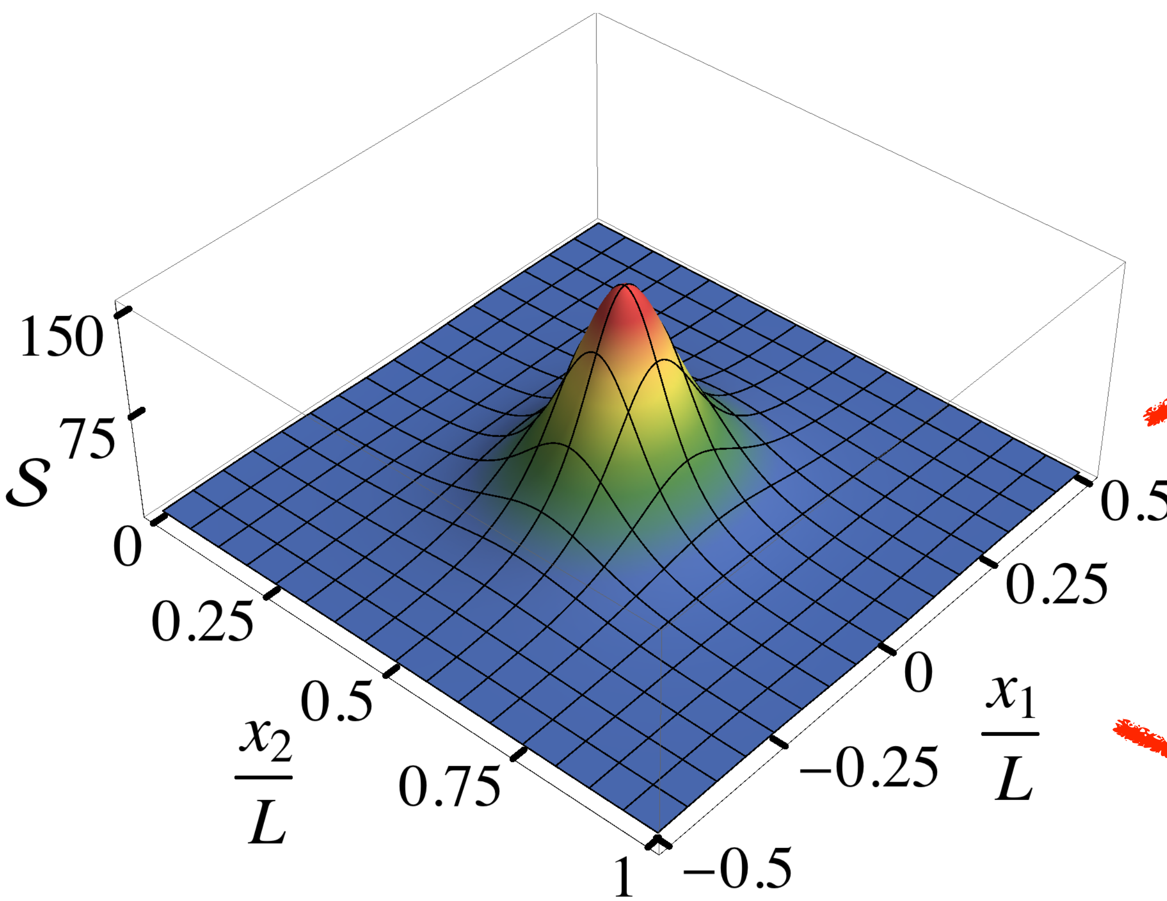
O(N) model: $\pi_2[O(N)] = 0$; SU(N) Principal chiral model $\pi_2[SU(N)] = 0$

The fractons, or composites built from them, drive appearance of mass gap!

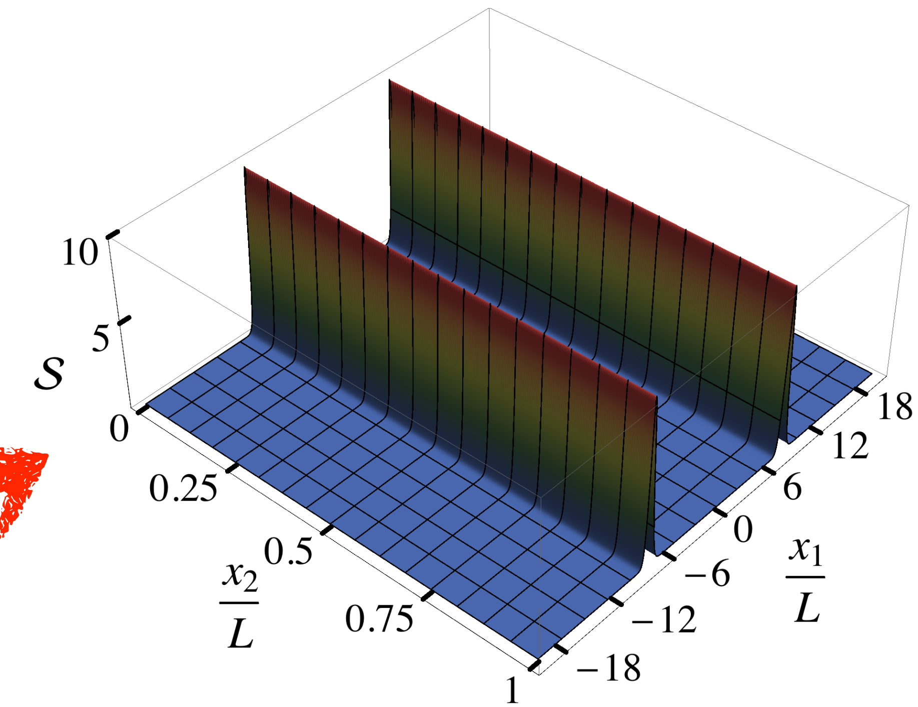
$$[\mathcal{F}] \sim e^{-\frac{c}{\lambda}}, \ \lambda = g^2 N, c \sim \mathcal{O}(1)$$

Fractionalization of unitons

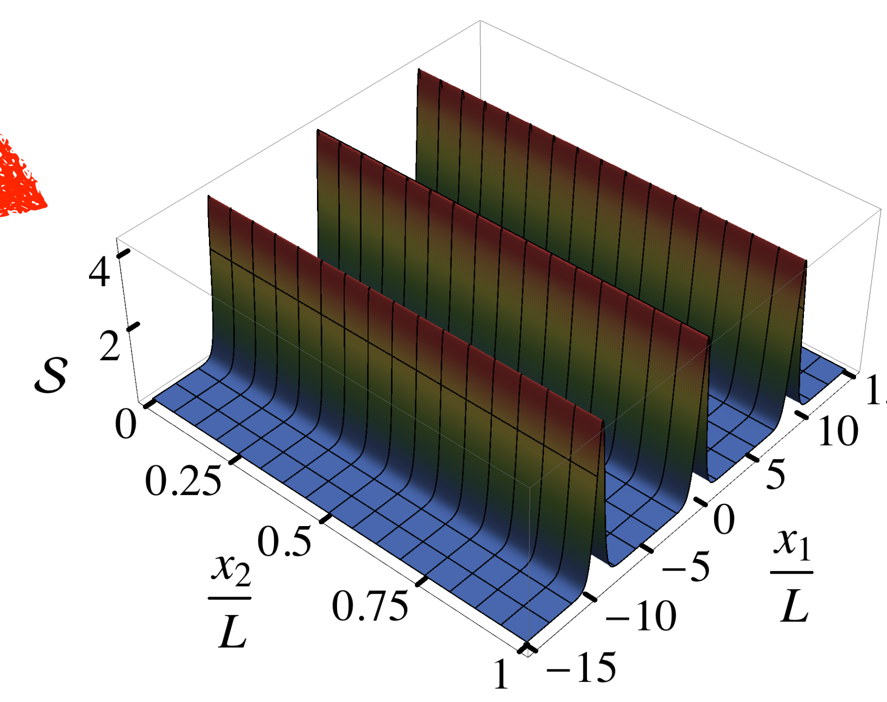
Uniton action density



Fracton action density



SU(2)



SU(3)

Resurgence in a box

To obtain results, use small NLΛ 1D effective field theory. EFT UV cutoff μ ∼ 1/(NL).

At small NLΛ, mass gap ends up looking like

$$\Delta = \mu\, e^{-\frac{c}{\lambda}} \left(\sum_n p_n \lambda^n + e^{-\frac{2c}{\lambda}} \sum_m b_m \lambda^m + \cdots \right)$$

Fracton ($\mathcal{F}$) effect

$\mathcal{F}\,\mathcal{F}\bar{\mathcal{F}}$ effect

Fluctuations

Schematic expression: really there's log(λ) factors, and sometimes gap starts at with contributions from two fractons, etc

The series appearing above are resurgent.

Resurgence in a box

AC, Dorigoni, Unsal
coming soon

So, seems resurgence applies to 2D QFTs — at least to leading order.

But the check used that small-L EFT, which is QM.

From the perspective of earlier worries, this is a bit of a cheat!

A demonstration directly in $d = 2$, without compactification, would be better.

Resurgence in full QFT

AC, Dorigoni, Unsal
coming soon

Warning: work in progress from here onward!

Use large N expansion to get around strong-coupling issues on R^2

Idea is to work perturbatively in 1/N, but exactly in 't Hooft coupling, then explore 't Hooft coupling expansion structure.

Example for this talk: 2D O(N) model

$$S = \int_{\mathbb{R}^2} d^2x \, \partial_\mu n_a \partial^\mu n^a \, , \; n_a n^a = \frac{N}{4\pi\lambda}, a = 1, \cdots , N.$$

Results generalize to other vector-like NLSMs

Resurgence in large N O(N) model

Integrate in a Lagrange multiplier σ to make life easier:

$$S = \int_{\mathbb{R}^2} d^2x \left[\partial_\mu n^a \partial^\mu n_a - \sigma \left(n^a n_a - \frac{N}{4\pi\lambda}\right)\right]$$

Questions: what's the mass gap Δ? Resurgence as a function of λ?

Perturbation theory: theory of N - 1 massless particles, Δ = 0.

To define theory, must regularize UV. We'll use momentum cutoff μ .

$$\frac{d\lambda}{d\log\mu} = -2\lambda^2\left[1+\frac{4}{N}\right] + \mathcal{O}\left(\frac{\lambda^3}{N}\right)$$

$$\Lambda_{\text{one-loop}} \sim \mu\, e^{-\frac{1}{2\lambda}}$$

Mass gap physics far outside any semiclassical regime on R^2!

Resurgence in large N O(N) model

Large N solution is textbook material - see e.g. Peskin & Schoeder

Integrate out n^a fields, giving

$$S = N \int_{\mathbb{R}^2} d^2x \left[\frac{\sigma}{4\pi\lambda} - \frac{1}{2} \mathrm{Tr}\ \log(\partial^2 - \sigma) \right]$$

At large N, physics captured by saddle-point for σ, which satisfies

$$\frac{\partial S}{\partial \sigma} = 0 \ \Rightarrow \ \int^{|p|<\mu} \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + \sigma} = \frac{1}{4\pi\lambda}$$

Want σ in terms of μ and λ.
Non-zero σ is a mass-squared for n^a fields!

Resurgence in large N O(N) model

$$\frac{\partial S}{\partial \sigma} = 0 \;\Rightarrow\; \int^{|p|<\mu} \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2 + \sigma} = \frac{1}{4\pi\lambda}$$

The textbooks all say that

$$\int^{|p|<\mu} \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2 + \sigma} = \frac{1}{4\pi} \log\left(\frac{\mu^2}{\sigma}\right)$$

$$\Rightarrow \quad \sigma = \mu^2 e^{-1/\lambda}$$

Spectrum has N massive particles, with $m^2 = \sigma$

$$\frac{\partial \sigma}{\partial \log \mu} = 0 \;\Rightarrow\; \frac{\partial \lambda}{\partial \log \mu} = -2\lambda^2$$

Celebrated result: O(N) beta function is one-loop exact at large N

Resurgence in large N O(N) model

Compare large N result on R² to adiabatic-small-L expectation:

$$\Delta = \mu \, e^{-1/2\lambda}$$

versus

$$\Delta = \mu \, e^{-\frac{c}{\lambda}} \left(\sum_n p_n \lambda^n + e^{-\frac{2c}{\lambda}} \sum_m b_m \lambda^m + \cdots \right)$$

Fracton ($\mathcal{F}$) effect

$\mathcal{F}\,\mathcal{F}\bar{\mathcal{F}}$ effect

Fluctuations

Large N limit suppresses fluctuations and kills multi-fractons!?

Conceivable… But is it true?

Resurgence in large N O(N) model

AC, Dorigoni, Unsal
coming soon

The textbooks all say that

$$\int^{|p|<\mu} \frac{d^2p}{(2\pi)^2}\,\frac{1}{p^2+\sigma} = \frac{1}{4\pi}\log\left(\frac{\mu^2}{\sigma}\right)$$

Bizarre fact: the equal sign is wrong.

$$\int^{|p|<\mu} \frac{d^2p}{(2\pi)^2}\,\frac{1}{p^2+\sigma} = \frac{1}{4\pi}\log\left(\frac{\mu^2+\sigma}{\sigma}\right)$$

Consequences:

$$\sigma = \mu\, e^{-1/\lambda}\,\frac{1}{1-e^{-1/\lambda}}$$

$$\frac{\partial\lambda}{\partial\log\mu} = -2\left(1-e^{-1/\lambda}\right)\lambda^2$$

non-perturbative corrections!

Coupling constant flow

AC, Dorigoni, Unsal coming soon

One-loop coupling diverges at $\mu = \Lambda = e^{-1/2\lambda}$:

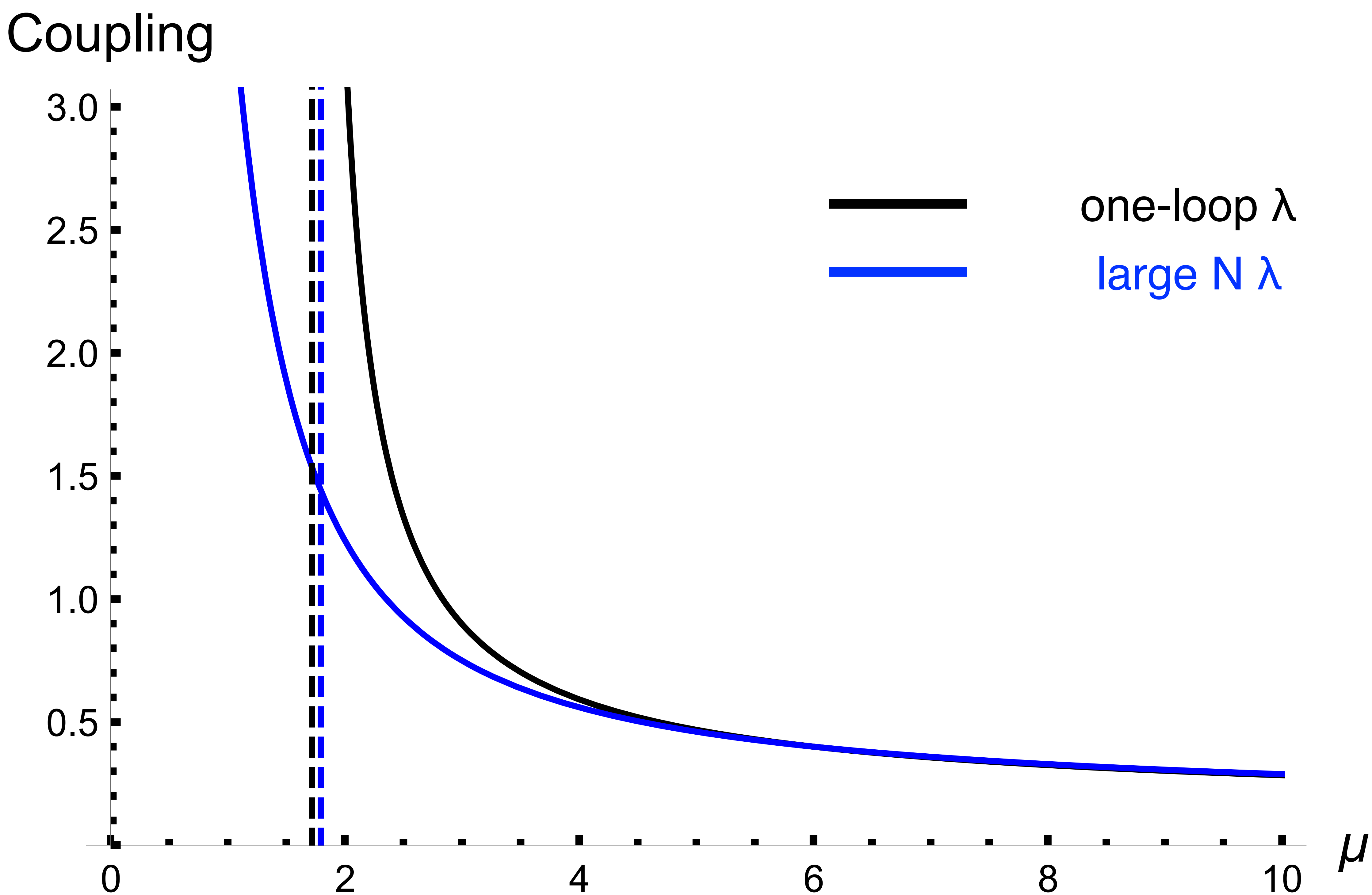
$$\lambda_{\mathrm{one-loop}}[\mu] = \frac{\lambda_0}{2\lambda_0 \log\left(\frac{\mu}{\mu_0}\right) + 1}$$

Exact large N coupling only diverges at $\mu = 0$:

$$\lambda(\mu) = \frac{1}{\log\left(1 + \frac{\mu^2}{\mu_0^2}\left(e^{+1/\lambda_0} - 1\right)\right)}$$

Coupling constant flow

AC, Dorigoni, Unsal coming soon



Resurgence in large N O(N) model

AC, Dorigoni, Unsal coming soon

Compare large N result on R^2 to adiabatic-small-L expectation:

R^2

$$\Delta\big|_{N=\infty} = \mu\, e^{-\frac{1}{2\lambda}} \left(1 + \frac{1}{2} e^{-\frac{2}{2\lambda}} + \frac{3}{8} e^{-\frac{4}{2\lambda}} + \cdots \right)$$

versus

Small L, $R \times S^1$, $N < \infty$

$$\Delta = \mu\, e^{-\frac{c}{\lambda}} \left(\sum_n p_n \lambda^n + e^{-\frac{2c}{\lambda}} \sum_m b_m \lambda^m + \cdots \right)$$

Fracton ($\mathcal{F}$) effect

$\mathcal{F}\mathcal{F}\bar{\mathcal{F}}$ effect

Fluctuations

Large N limit still suppresses fluctuations; but way closer resemblance!

Are the `fractons' somehow surviving all the way to strong coupling?

Exact large N mass gap & coupling

AC, Dorigoni, Unsal coming soon

We're still confused on what to make of all this.

Well known that only first two coefficients of beta functions invariant under scheme changes.

More precisely, first two coefficients of series expansion of beta function invariant under scheme changes represented by power series.

Still trying to understand whether any extra \`non-perturbative universality' can be revealed by trans-series perspective.

In any case, tantalizing that exact large N result has some interesting properties + resonance with small-L studies.

O(N) model at large N

AC, Dorigoni, Unsal
coming soon;
also F. David 1984

So far, we have a transseries but no resurgence, due to suppression of fluctuations by large N

$$\Delta|_{N=\infty} = \mu\, e^{-\frac{1}{2\lambda}} \left(1 + \frac{1}{2} e^{-\frac{2}{2\lambda}} + \frac{3}{8} e^{-\frac{4}{2\lambda}} + \cdots \right)$$

To see resurgent behavior, need to look at 1/N corrections.

To be specific, we'll continue to examine < σ >

$$\langle \sigma \rangle = \left\langle \frac{4\pi\lambda}{N} \partial_\mu n_a \partial^\mu n^a \right\rangle = \Delta^2$$

O(N) model at order 1/N

AC, Dorigoni, Unsal
coming soon;
also F. David 1984

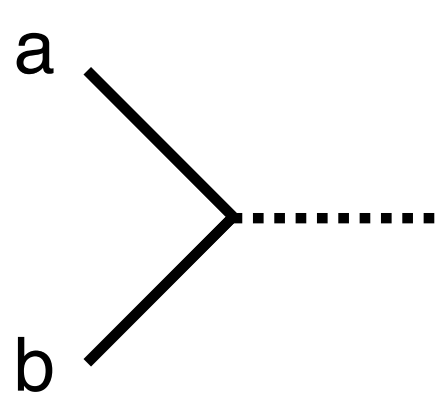
Large N theory consists of N massive fields with mass m = Δ

a b $$G^{ab}(p) = \frac{\delta^{ab}}{p^2 + m^2}$$

and a field \`σ’ describing fluctuations around VEV, $\sigma \to \langle \sigma \rangle + \sigma/N^{1/2}$

$$G_\sigma(p) = \frac{-4\pi\sqrt{p^2(p^2+4m^2)}}{\log\left[\frac{\sqrt{p^2+4m^2}+\sqrt{p^2}}{\sqrt{p^2+4m^2}-\sqrt{p^2}}\right]}$$

with an interaction vertex



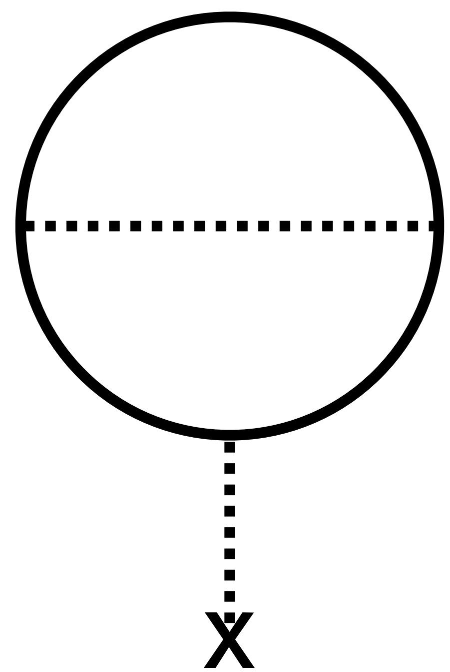
$$\frac{1}{\sqrt{N}}\delta^{ab}$$

Dependence on λ only enters through m!

O(N) model at order 1/N

AC, Dorigoni, Unsal
coming soon;
also F. David 1984

Leading correction to < σ > comes from



$$\langle \sigma \rangle = m^2 + \frac{1}{N} I(\mu, m) + \mathcal{O}(1/N^2)$$

The 1/N correction is UV-divergent. Put cutoff at μ, assume μ ~ N^0

$$I(\mu, m) = \frac{1}{2} G_\sigma(0) \int^{|p| \le \mu} \frac{d^2 p}{(2\pi)^2} \int \frac{d^2 k}{(2\pi)^2} G_\sigma(p) G_{ab}(k) G^{bc}(k) G^a_c(p+k).$$

O(N) model at order 1/N

AC, Dorigoni, Unsal
coming soon;
also F. David 1984

Evaluating the integrals, get ugly but (eventually!) instructive result:

$$I(\mu, m) = m^2 \left(-E_i \left[\frac{1}{2} \log A(\mu, m) \right] + E_1 \left[\frac{1}{2} \log A(\mu, m) \right] + 2\gamma_E + 2 \log \left[\frac{1}{2} \log A(\mu, m) \right] - 2 \log \left[1 + \frac{\mu^2}{4m^2} \right] \right)$$

$$A(\mu, m) = \left(\sqrt{1 + \frac{\mu}{4m^2}} + \sqrt{\frac{\mu}{4m^2}} \right)^4$$

The 1/N correction is entirely unambiguous at this stage. Statement almost trivial: Given a regulator, path integral will be unambiguous.

Where's the resurgence?

O(N) model at order 1/N

AC, Dorigoni, Unsal
coming soon;
also F. David 1984

Interested in resurgence properties in λ - so note that

$$\frac{1}{\tilde{\lambda}} \equiv \frac{1}{2}\log A(\mu, m) = \frac{1}{\lambda} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{c_n}{n}e^{-n/\lambda}$$

`central trinomial coefficients';
series converges.

Expansions of the exponential-integral functions in λ are asymptotic:

$$E_i\left(\frac{1}{\tilde{\lambda}}\right) = \begin{cases} -i\pi + e^{1/\tilde{\lambda}}\sum_{n=0}^{\infty} n!\tilde{\lambda}^{n+1}\,, & 0 < \arg(\tilde{\lambda}) < \pi\,, \\ +i\pi + e^{1/\tilde{\lambda}}\sum_{n=0}^{\infty} n!\tilde{\lambda}^{n+1}\,, & -\pi < \arg(\tilde{\lambda}) < 0\,, \end{cases}$$

$$E_1\left(\frac{1}{\tilde{\lambda}}\right) = e^{-1/\tilde{\lambda}}\sum_{n=0}^{\infty}(-1)^n n!\tilde{\lambda}^{n+1}\,.$$

O(N) model at order 1/N

AC, Dorigoni, Unsal
coming soon;
also F. David 1984

Plug these expansions back into < σ >, to find

$$I(\mu,\lambda) \simeq \mu^2 \left(\sum_{n=0} n!\lambda^{n+1} \mp i\,\pi\, e^{-\frac{1}{\lambda}} + \cdots \right)$$

Factorial growth leads to renormalon ambiguity, which is cancelled by non-perturbative contribution.

Working out the …'s, we find that full expression at order 1/N indeed takes form of resurgent transseries.

Results strongly support idea that observables in asymptotically-free theories on R^2 are given by resurgent transseries in λ!

O(N) model at order 1/N

AC, Dorigoni, Unsal
coming soon;

At this point you could ask, if < σ > = m^2 + I(μ,λ)/N + …, and

$$I(\mu,\lambda) \simeq \mu^2 \left(\sum_{n=0} n!\lambda^{n+1} \mp i\,\pi\, e^{-\frac{1}{\lambda}} + \cdots \right)$$

(1) What happens if we subtract `all' divergences?
Does < σ > then become ambiguous?

Find that counter-terms pick up ambiguities,
but < σ > stays unambiguous.

(Still working on better understanding of this all-orders renormalization.)

(2) If dim-reg is used, no power divergences. Ambiguous result?

F. David
1984: yes.

No. "Dimensional regularization" is not
a valid regulator non-perturbatively.

Dimensional regularization

Idea of dim-reg:

$$\int \frac{d^d p}{(2\pi)^d} \frac{(p^2)^a}{(p^2+m^2)^b} \longrightarrow \mu^{d-n} \int \frac{d^n p}{(2\pi)^n} \frac{(p^2)^a}{(p^2+m^2)^b}$$

(1) Find `n' where integral from |p|=0 to |p| = ∞ converges, then do it:

$$\frac{1}{(4\pi)^{d/2}} \frac{\mu^{d-n}}{\Delta^{b-a-d/2}} \frac{\Gamma(a+d/2)\Gamma(b-a-d/2)}{\Gamma(b)\Gamma(d/2)}$$

(2) Expand near desired dimension d, discard poles like 1/(n-d) = 1/ϵ

(3) Profit from remaining $\log(m^2/\mu^2)$ terms!

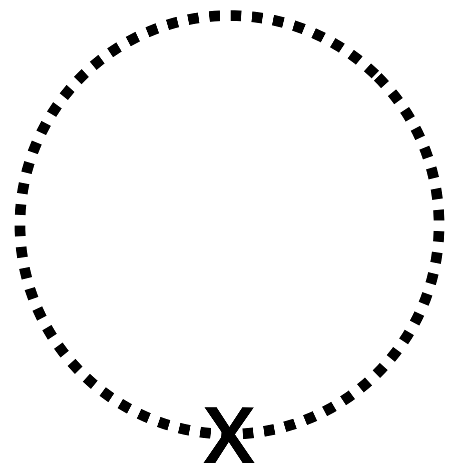
No explicit power divergences.

Recipe works to any fixed order in perturbation theory.

Failure of dimensional regularization

In the large N O(N) model, dim-reg fails at step 1. Example:

$$\langle \sigma^2 \rangle - m^4 =$$



$$= \frac{1}{N} G_\sigma(0)^2 \int \frac{d^2 p}{(2\pi)^2} G_\sigma(p)$$

$$p \to \infty, \; G_\sigma \sim \frac{p^2}{\log(p^2/m^2)}$$

$$p \to 0, \; G_\sigma \sim m^2$$

(Using $G_\sigma(p,n)$ doesn't help!)

In dimension n, need Re[n] < -3 in UV and Re[n] > 0 in IR for convegence.

No choice of n gives finite result.
Dimensional `regularization' is not a regulator non-perturbatively.

Perhaps not so shocking, but amusing to see explicit illustration.

Conclusions

Not obvious that resurgence should apply in $d > 1$.

But it **does**, as illustrated using large N solution of 2D models!

"We know much more than we can prove…"

Peculiarity of vector-type models - need 1/N effects to see resurgence.
Expect resurgence at leading order in matrix-type theories.

Mass gap Δ on R^2 has close resemblance to adiabatic-small-L Δ

Large N β-function of 2D sigma models is **not** one-loop exact - there are non-perturbative corrections.

Regularization is subtle at non-perturbative level.
Dimensional regularization isn't regularization.

Privileged role for explicit cut-off regulators?

The end