Lecture VI: Neutrino propagator and neutrino potential

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For the case we are considering, i.e. with the exchange of light Majorana neutrinos, the double beta decay nuclear matrix element consists of three parts:

$$M^{0\nu} = M_{\rm GT}^{0\nu} - \frac{M_F^{0\nu}}{g_A^2} + M_T^{0\nu} \equiv M_{\rm GT}^{0\nu}(1 + \chi_F + \chi_T),$$

The Gamow-Teller part M_{GT} is the dominant one. When treated in the closure approximation it is

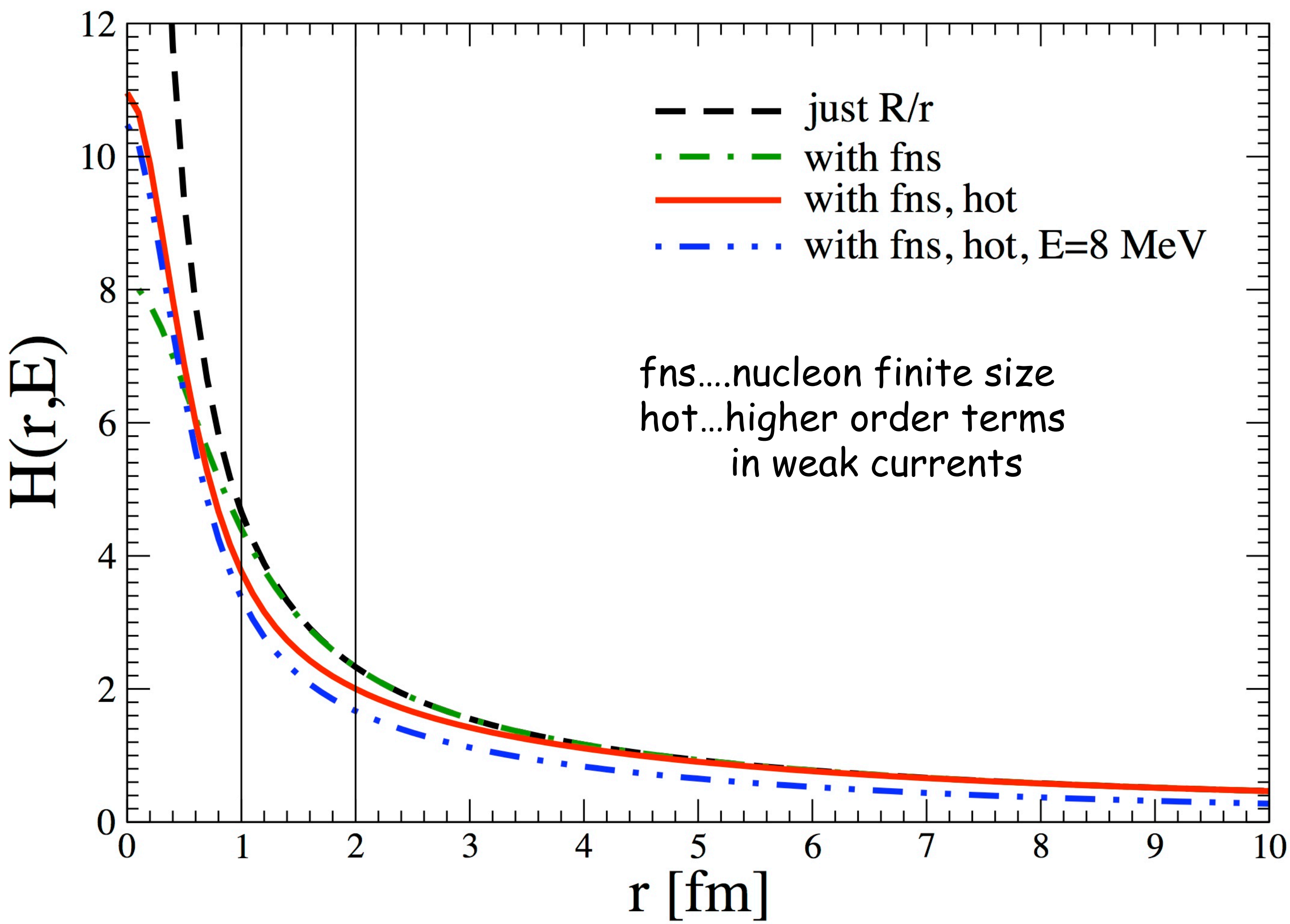
$$M_{\rm GT}^{0\nu} = \langle f | \Sigma_{lk} \sigma_l \cdot \sigma_k \tau_l^+ \tau_k^+ H(r_{lk}, \bar{E}) | i \rangle,$$

The ``neutrino potential" originating from the light neutrino propagator is

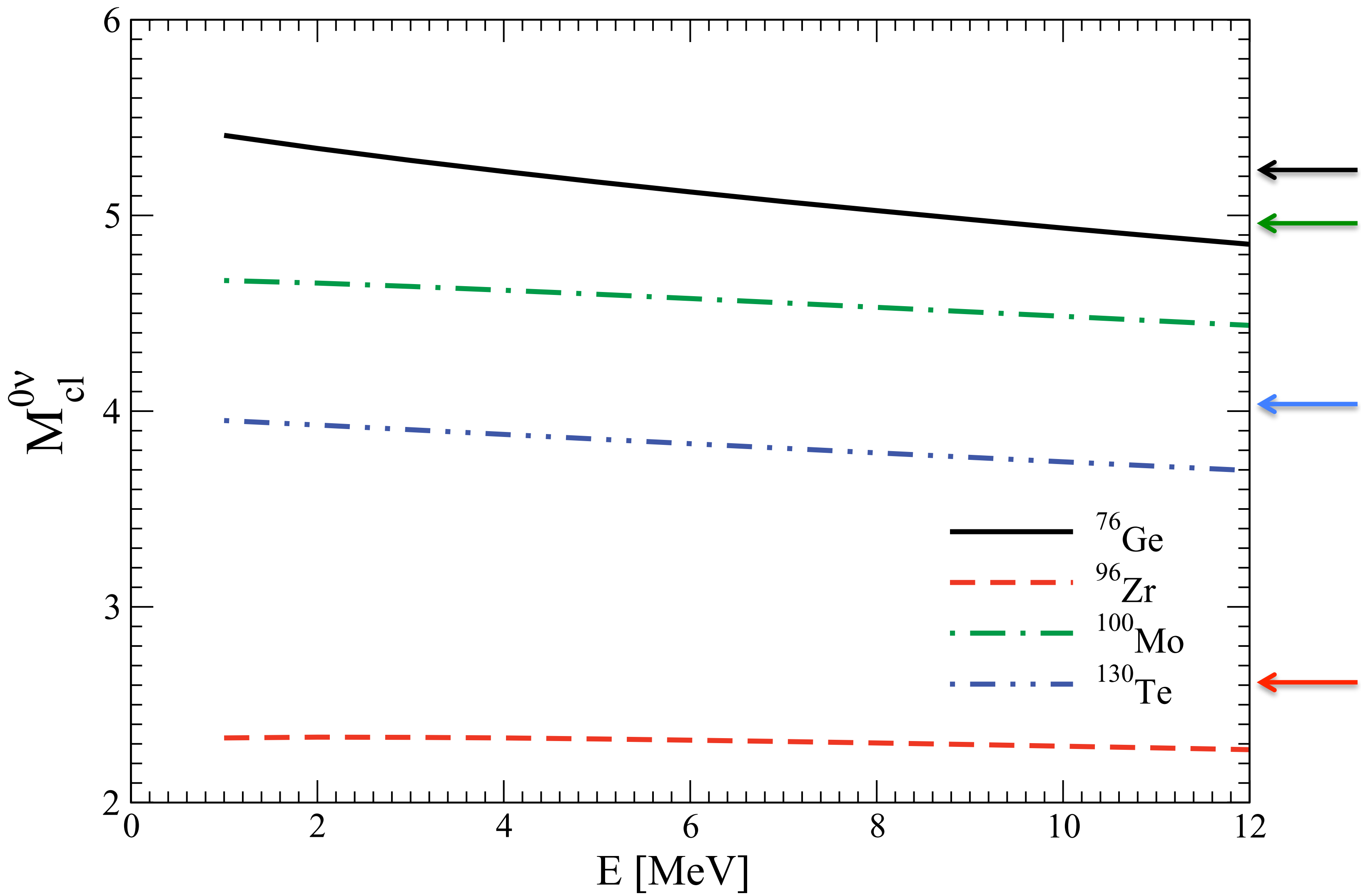
$$H_K\left(r_{12}, E_{J^\pi}^k\right) = \frac{2}{\pi g_A^2} R \int_0^\infty f_K(q r_{12}) \frac{h_K(q^2) q dq}{q + E_{J^\pi}^k - (E_i + E_f)/2}.$$

Where $f_{GT}(qr) = j_0(qr)$ and $h_{GT} = g_A/(1 + q^2/M_A{}^2)^2$ is the nuclear axial current form factor, $M_A \sim 1$ GeV.

As we will see, the neutrino momentum is ~200 MeV so the dependence on the nuclear excitation energy is weak. The potential H(r,E) looks like a Coulomb 1/r radial dependence. Finite size and higher order currents remove the singularity at r=0.



Matrix elements $M^{0\nu}$ evaluated in closure approximation using the QRPA method. Plotted against assumed average excitation energy. Values without the closure approximation indicated by arrows. Closure approximation underestimates $M^{0\nu}$ by less than 10%.



How does the matrix element $M^{0\nu}{}_{GT}$ depend on the distance between the two neutrons that are transformed into two protons ? This is determined by the function $C^{0\nu}{}_{GT}(r)$

$$C^{0\nu}_{\rm GT}(r) = \langle f|\Sigma_{lk}\sigma_l \cdot \sigma_k \tau_l^+ \tau_k^+ \delta(r - r_{lk}) H(r_{lk}, \bar{E})|i\rangle,$$

It is normalized by the obvious relation $$M^{0\nu}_{\rm GT} = \int_0^\infty C^{0\nu}_{\rm GT}(r)\, dr,$$

Thus, if we could somehow determine C(r) we could obtain $M^{0\nu}$.

In order to obtain C(r) consider first the matrix elements of the operator $\sigma_1 \cdot \sigma_2$ between two neutrons and two protons coupled to the angular momentum J without the neutrino potential:

$$f^{\mathcal{J}}_{n,n',p,p'}(r) = \langle p(1),\, p'(2)(r); \mathcal{J} \parallel \boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2} \parallel n(1), n'(2)(r); \mathcal{J}\rangle$$

Here are few examples for the $f_{7/2}$ and $f_{5/2}$ orbits. These functions, as expected, typically extend up to the nuclear diameter, peaking near the middle. Some of them, in particular those with J = 0, are asymmetric with larger amplitude at small distances.

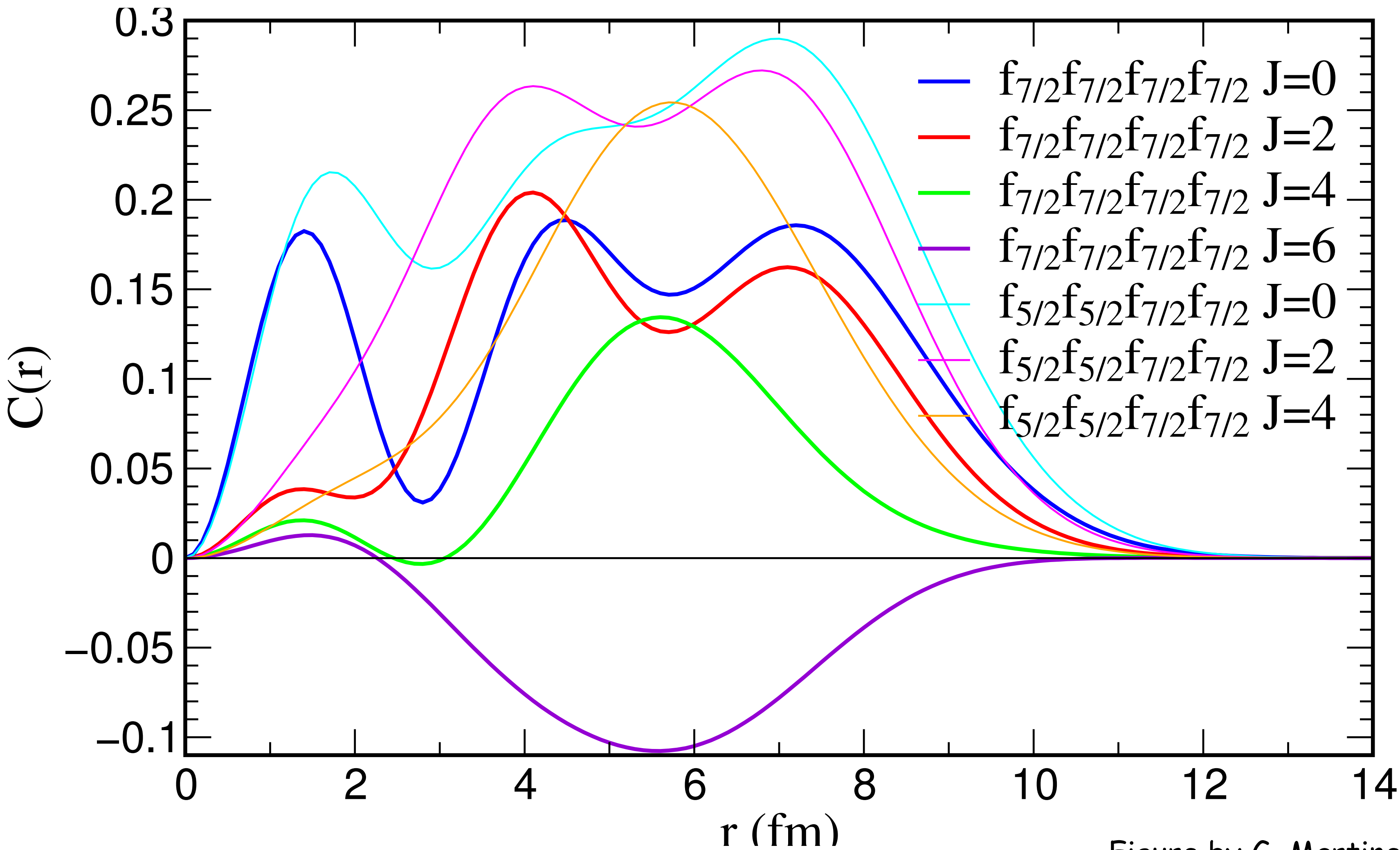
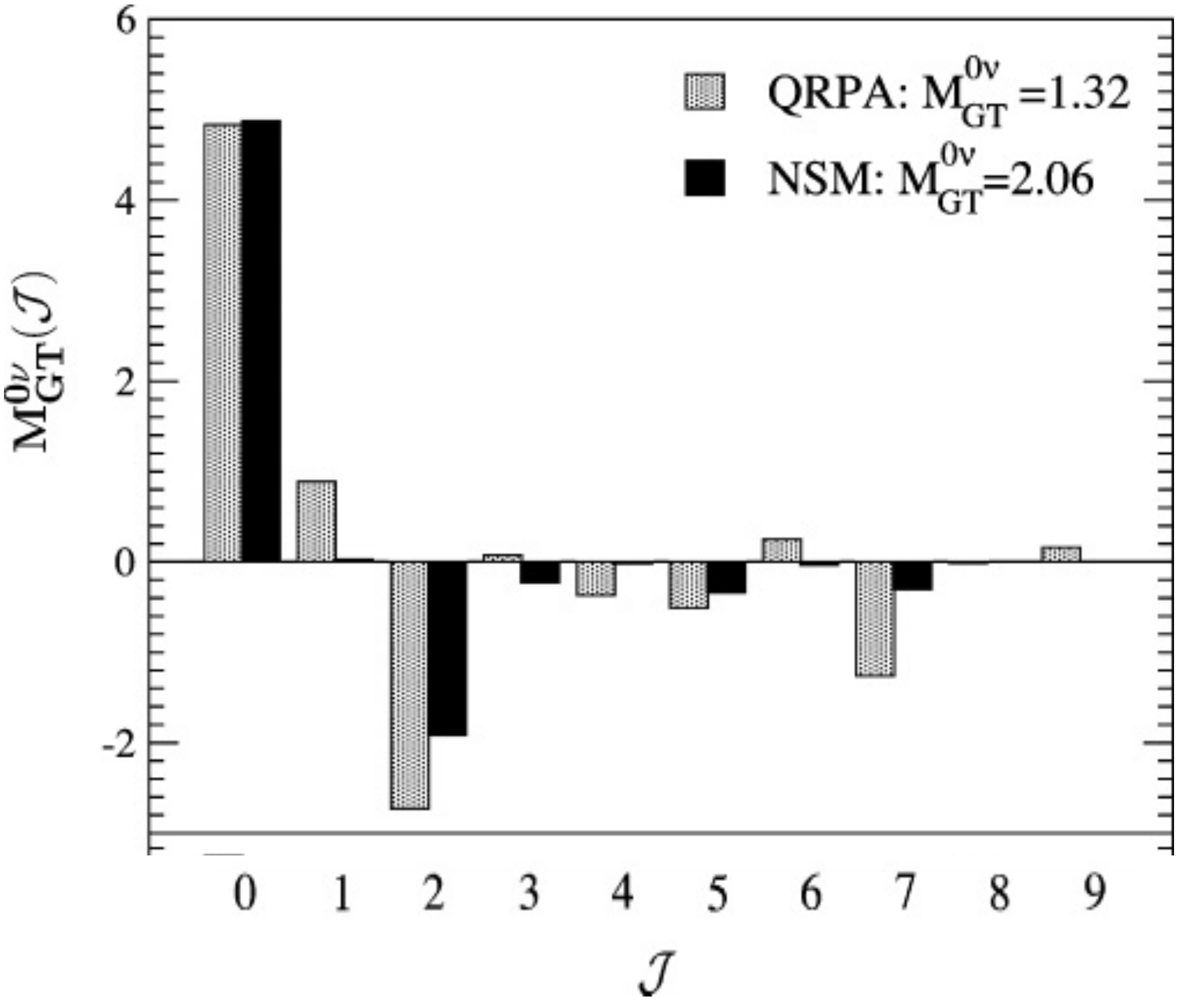


Figure by G. Martinez

To obtain the matrix element $M^{0\nu}$, one has to include the `neutrino potential' that stresses smaller values of r, and combine the s.p. states based on their contributions obtained by solving the corresponding equations of motion. In QRPA they are the amplitudes $\langle J^\pi k_i||[c_p^+ \tilde{c}_n]_J||0_i^+\rangle$ and $\langle 0_f^+||[c_{p'}^+ \tilde{c}_{n'}]_J||J^\pi k_f\rangle$

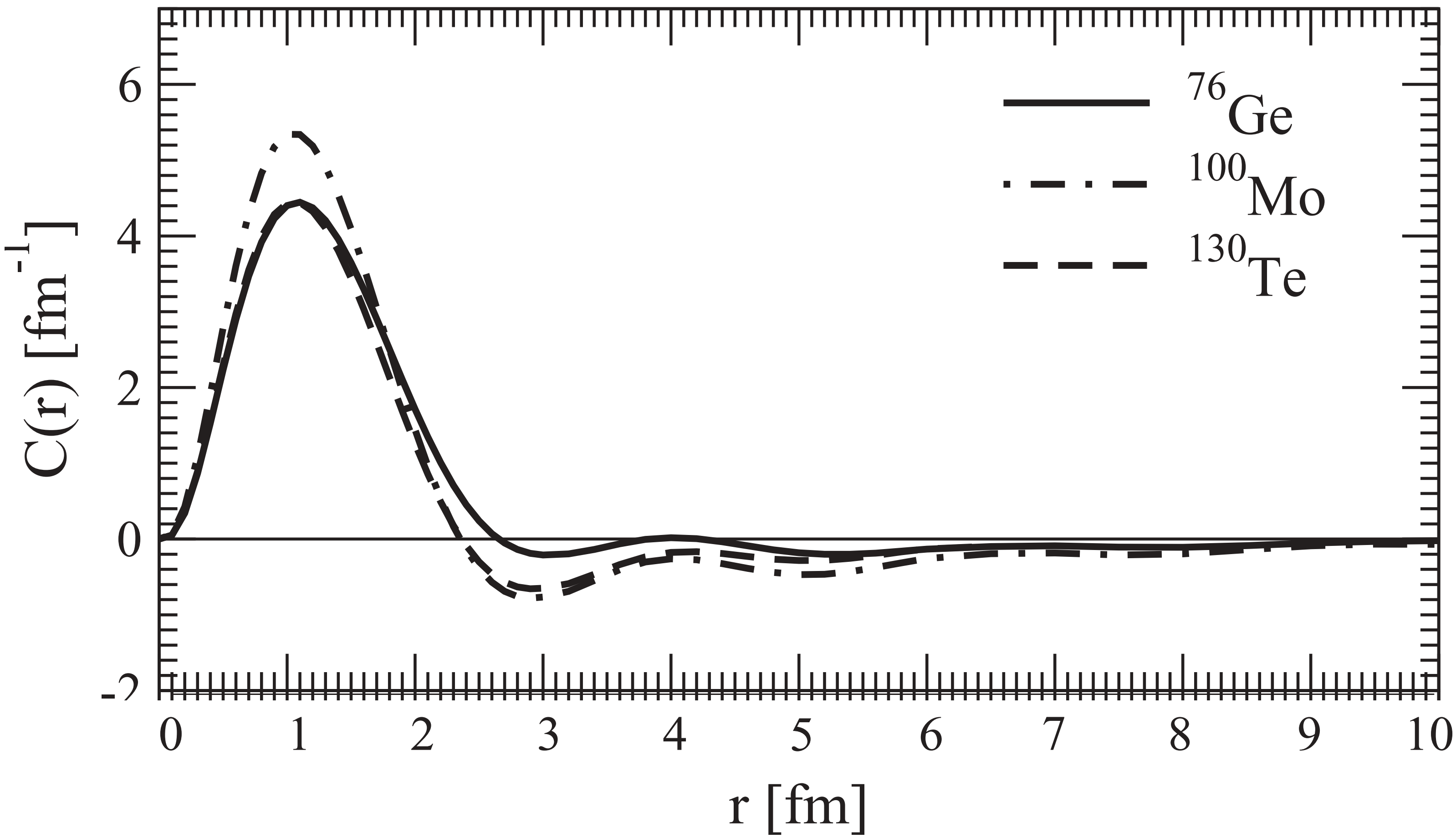
$$
\begin{aligned}
M_K = & \sum_{J^\pi, k_i, k_f, \mathcal{J}} \sum_{pnp'n'} (-1)^{j_n + j_{p'} + J + \mathcal{J}} \sqrt{2\mathcal{J}+1} \begin{Bmatrix} j_p & j_n & J \\ j_{n'} & j_{p'} & \mathcal{J} \end{Bmatrix} \\
& \times \langle p(1), p'(2); \mathcal{J} \| \bar{f}(r_{12}) O_K \bar{f}(r_{12}) \| n(1), n'(2); \mathcal{J} \rangle \\
& \times \langle 0_f^+ || \widetilde{[c_{p'}^+ \tilde{c}_{n'}]_J} || J^\pi k_f \rangle \langle J^\pi k_f | J^\pi k_i \rangle \\
& \times \langle J^\pi k_i || [c_p^+ \tilde{c}_n]_J || 0_i^+ \rangle . \qquad (4)
\end{aligned}
$$

It is instructive to consider the contributions of different angular momenta $\mathcal{J}$ to the final result. This is a typical case; $\mathcal{J}$ = 0 contributes most, while other $\mathcal{J}$ have smaller amplitude but opposite sign; hence a substantial cancellation. Note the qualitative agreement between NSM and QRPA.



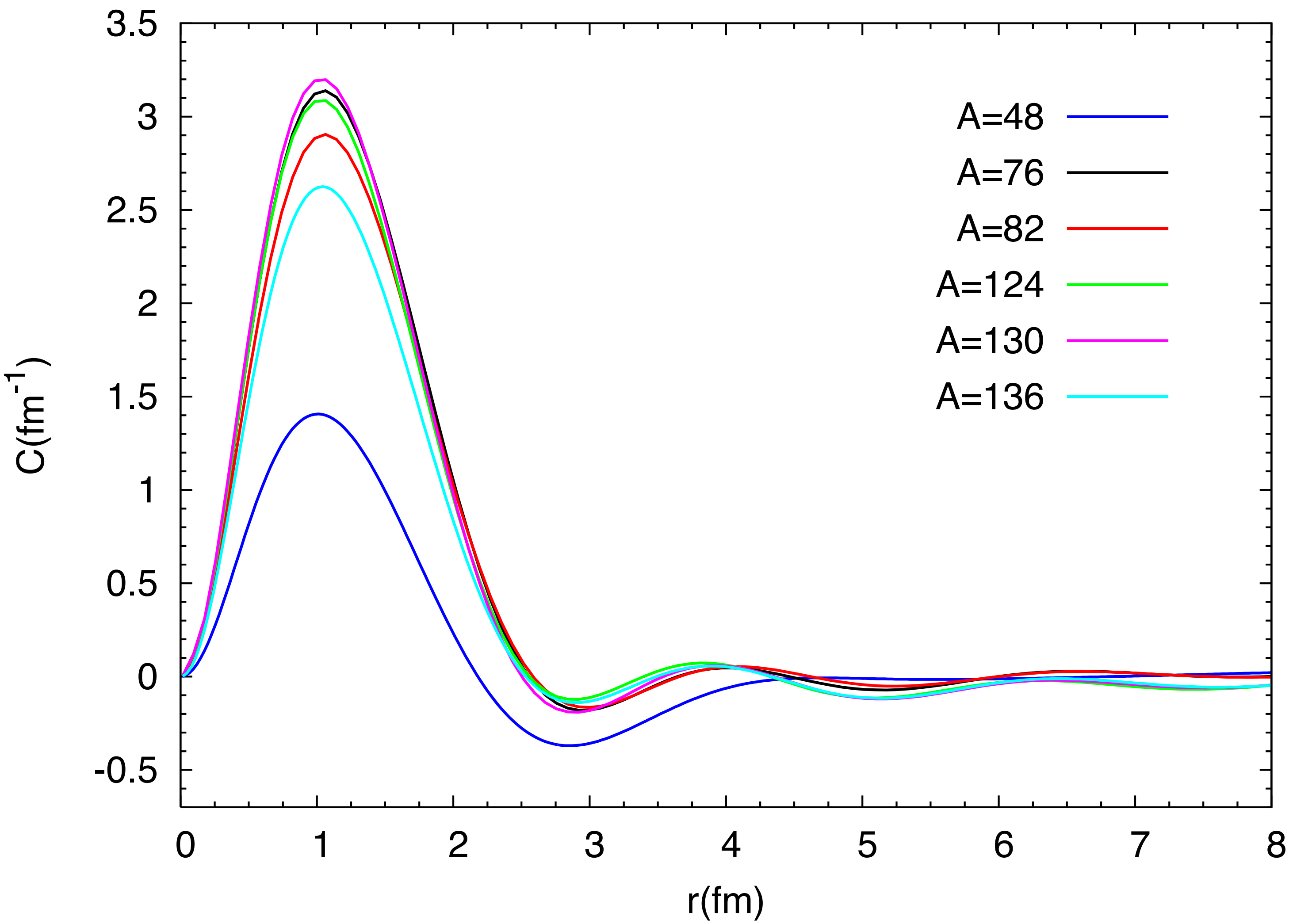
This is for the ^{82}Se. The same s.p. space, $f_{5/2}$, $p_{3/2}$, $p_{1/2}$, $g_{9/2}$, is used for both. For QRPA this space is smaller than usual, thus smaller $M^{0\nu}$ is obtained.

Function $C^{0\nu}(r)$ evaluated in QRPA. Note the peak at ~ 1fm. There is no contribution from r> 2-3 fm. And the function for different nuclei look very similar, essentially universal.



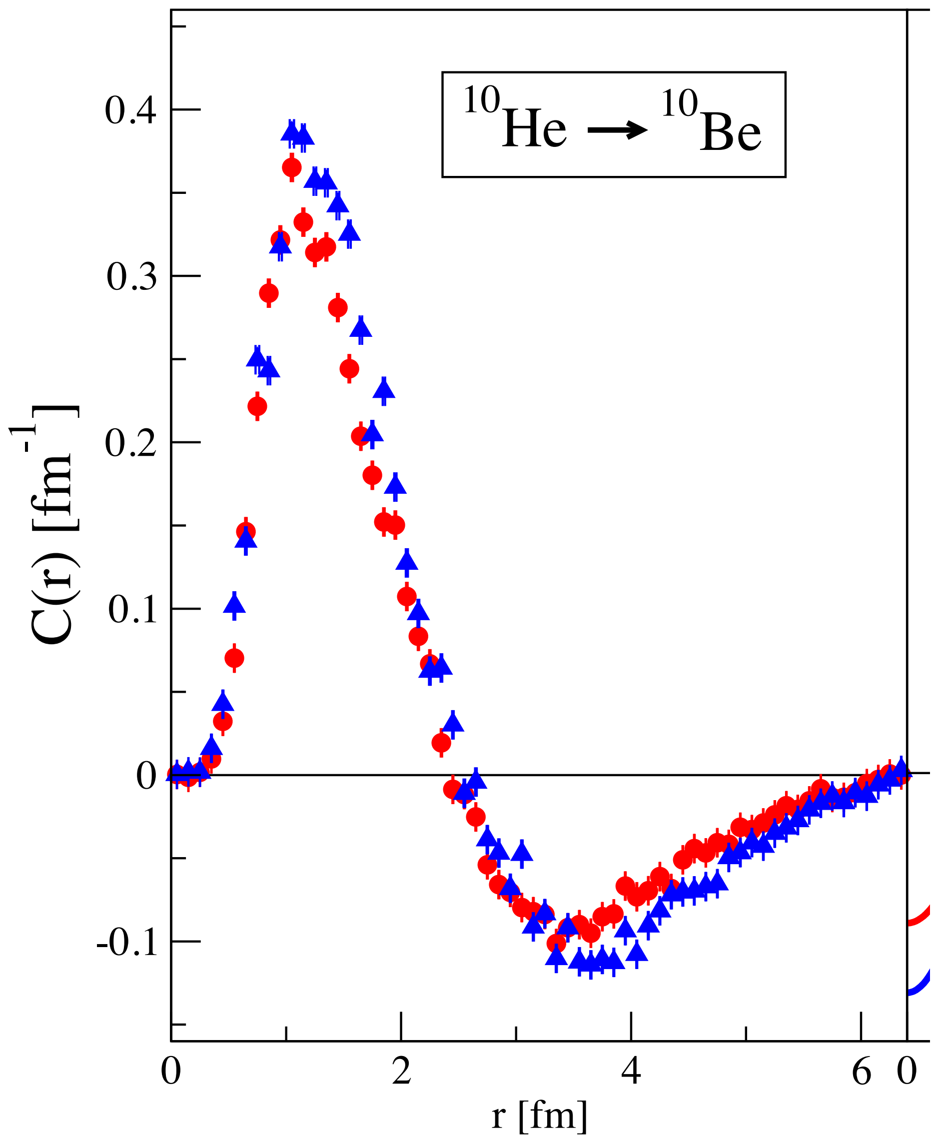
From Simkovic et al, Phys. Rev *C***77**, 045503 (2008)

Now C(r) evaluated in the nuclear shell model. All relevant features look the same as in QRPA despite the very different way the equations of motion are formulated and solved.



From Menendez et al, Nucl. Phys. *A* **818**, 130 (2009)

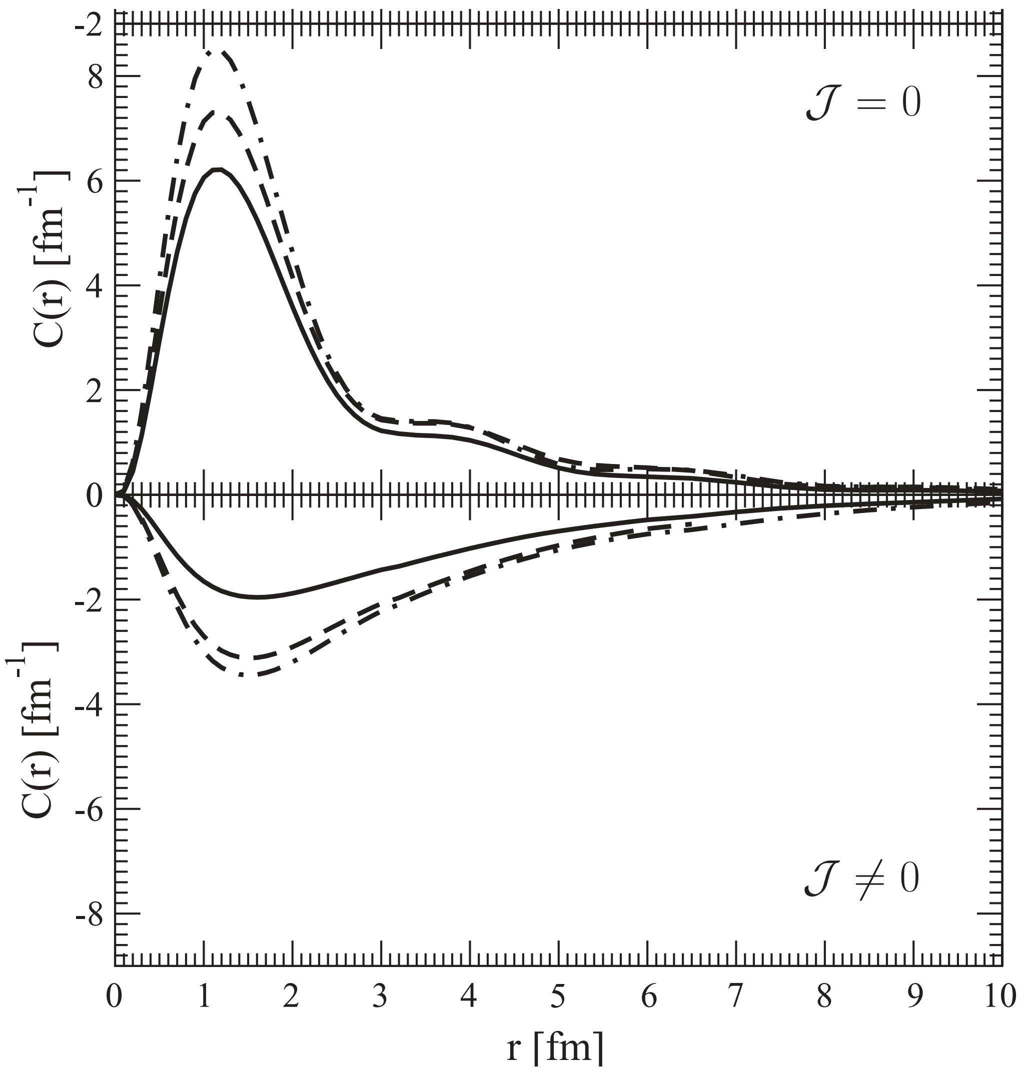
C(r) for the hypothetical $0\nu\beta\beta$ decay of ^{10}He.

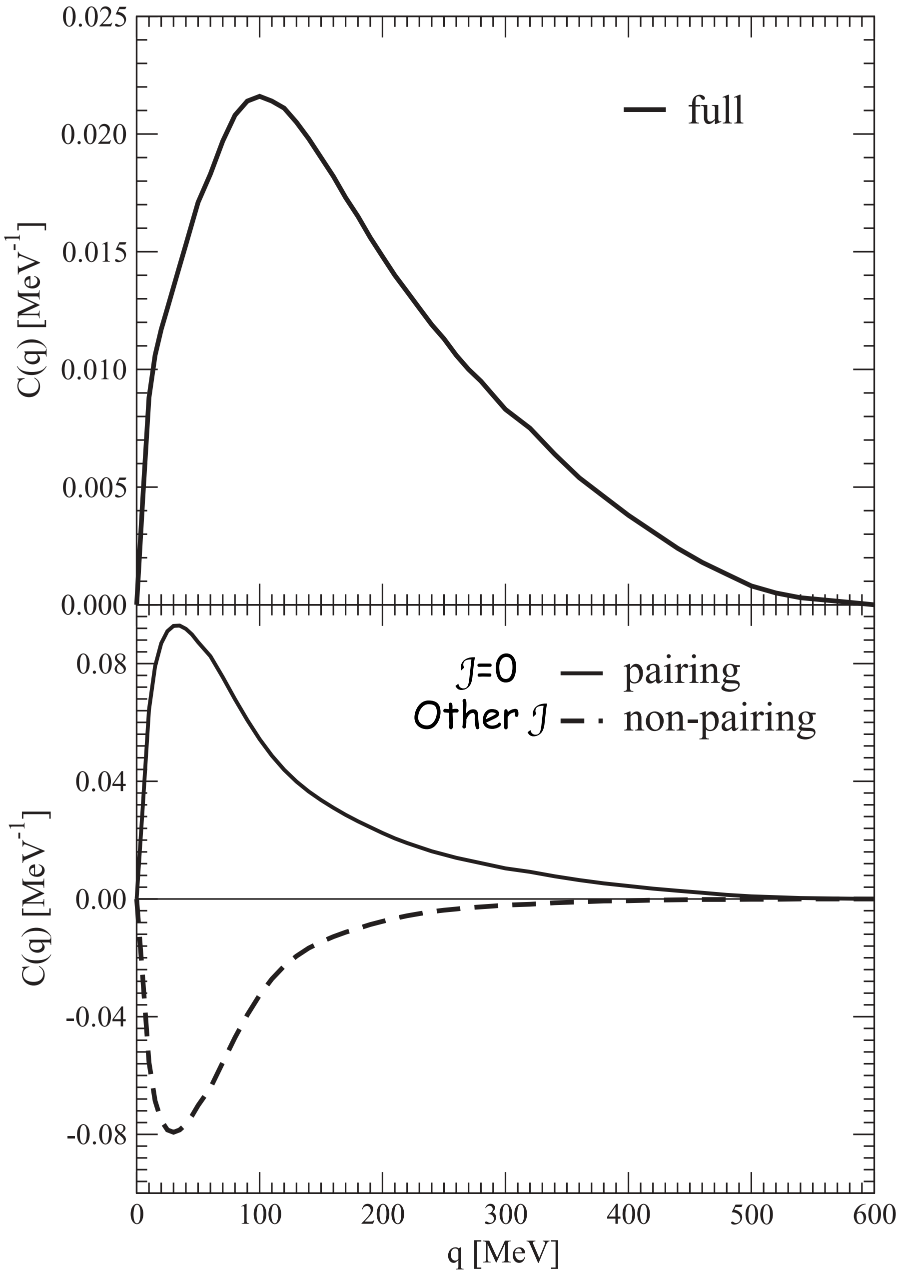


The calculation was performed using the *ab initio* variational Monte-Carlo method. So most of the approximations inherent in NSM or QRPA are avoided. Yet the C(r) function looks, at least qualitatively, very similar to the results shown before.

Figure from Pastore et al.,1710.05026

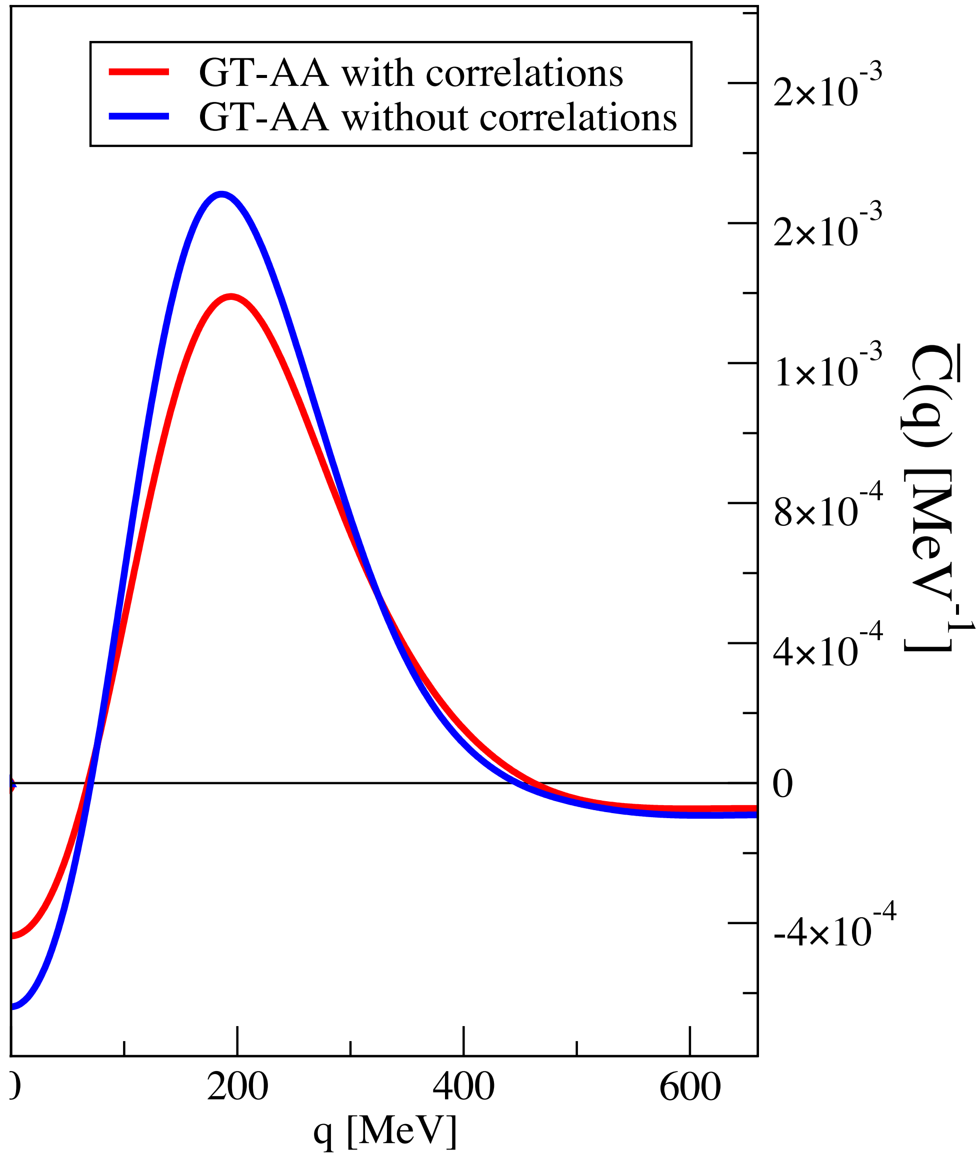
The fact that the resulting C(r) is concentrated at r<~2fm is the result of cancellation between $\mathcal{J}$ = 0 and other values of $\mathcal{J}$. We have seen the effect of such cancellation before. It is again common in QRPA and NSM.





From C(r) we know that the $2\nu\beta\beta$ operator has a short range character. That is also visible in the momentum analog C(q). characteristic momentum is not $\hbar c/R$ but $\hbar c/r_0 \sim 200$ MeV.

This is again the result of cancellation between the $\mathcal{J}$=0 (pairing) part and the other $\mathcal{J}$ (broken pairs) parts. Note that the lower panel has ~ 3 times larger y scale.



Again, $C(q)$ for the hypothetical ^{10}He $0\nu\beta\beta$ decay, evaluated using the variational Monte Carlo method, with no approximation. The behavior at large values of q (q > 400 MeV) is a bit different. This has to do with the different treatment of the nucleon finite size.

The short range character of the $0\nu\beta\beta$ operator, revealed by the evaluation of $C^{0\nu}(r)$ means that the nucleons participating in the decay must be close to each other. That also means that they are mostly in the central region of the nucleus, and less likely near the nuclear surface. The central regions of all nuclei has essentially the same density and thus also Fermi momentum, i.e. it is in the form of nuclear matter. It is thus not surprising that no matter which method is used there is relatively little a dependence in the $M^{0\nu}(Z,A)$.

This is in contrast with the known $M^{2\nu}$ matrix elements for the $2\nu\beta\beta$ decay which show a rather pronounce Z,A variations, $2\nu\beta\beta$ decay is low momentum transfer process, while the $0\nu\beta\beta$ Is much higher momentum transfer process.

Calculated $M^{0\nu}$ by different methods (color coded)
The spread of the M^{0n} values for each nucleus is ~ 3. On the other hand, there is relatively little variation from one nucleus to the next.

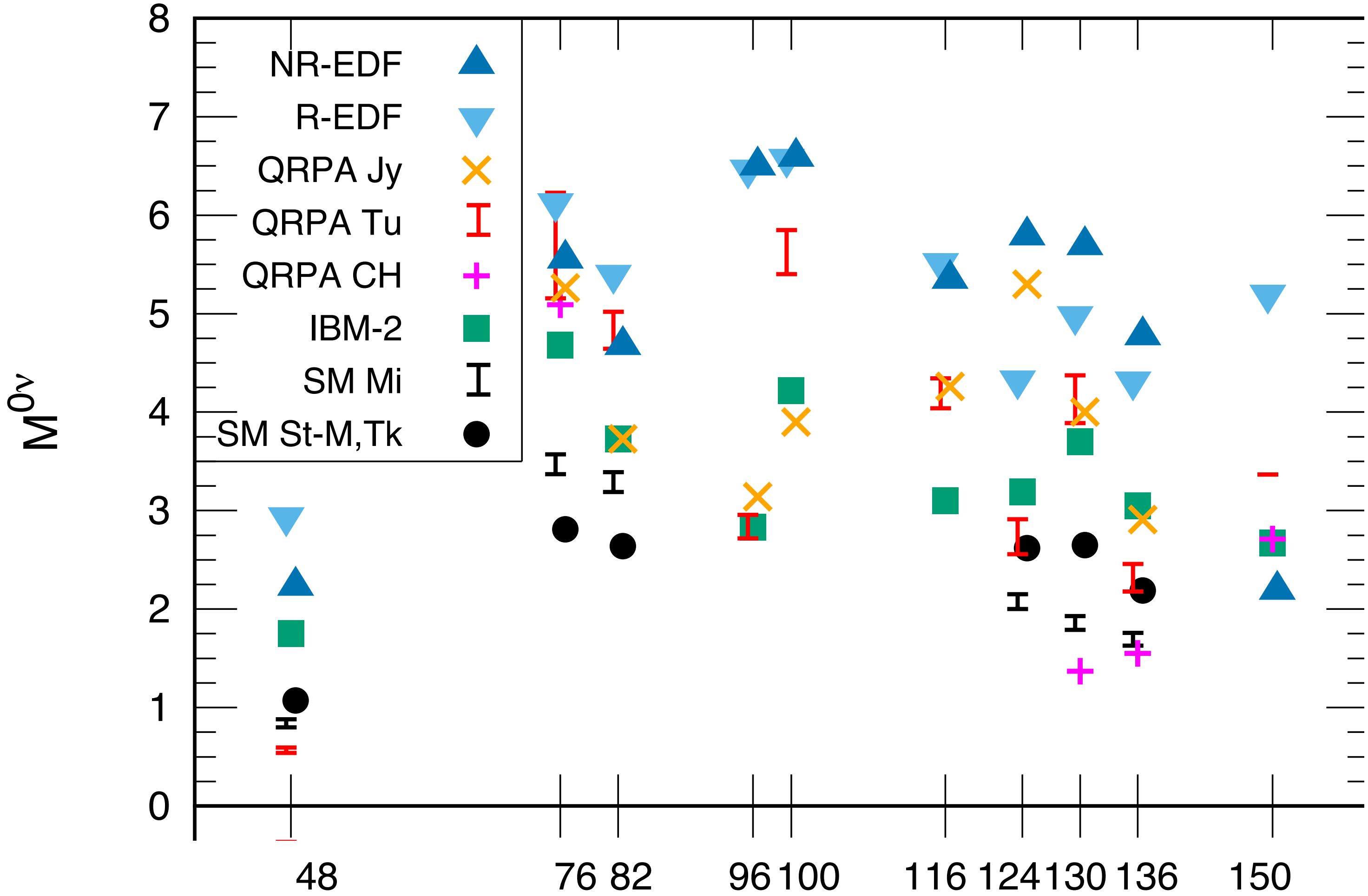
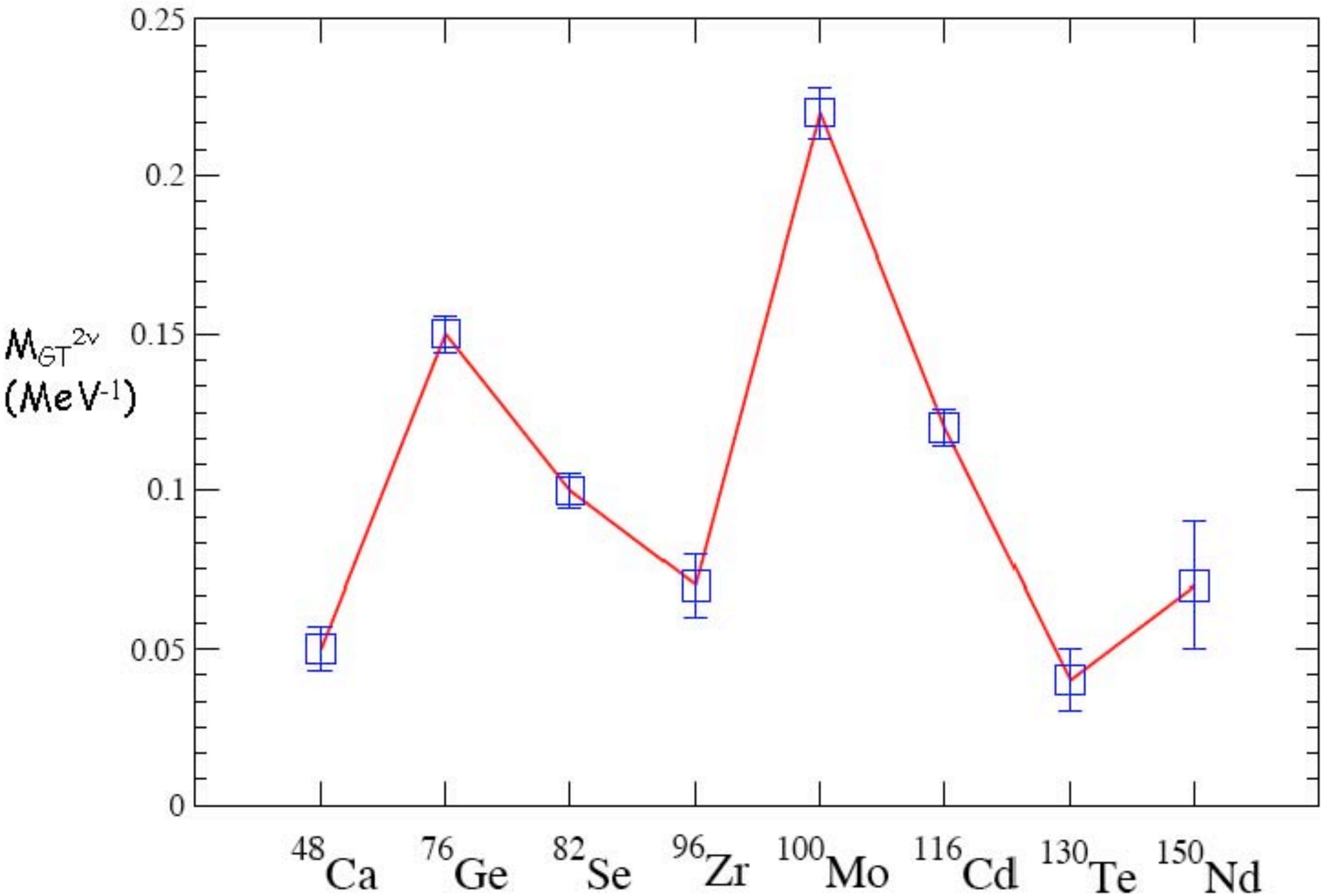


Figure from review by Engel and Menendez

The 2ν matrix elements, unlike the 0ν ones, exhibit pronounced shell effects. They vary relatively fast as a function of Z or A.



Lets consider once more the GT m.e. for $0\nu\beta\beta$

$$M_{\rm GT}^{0\nu} = \langle f | \Sigma_{lk} \sigma_l \cdot \sigma_k \tau_l^+ \tau_k^+ H(r_{lk}, \bar{E}) | i \rangle,$$

If we remove from the operator the neutrino potential H(r,E) we obtain the matrix element of the double GT operator connecting the ground states of the initial and final nuclei. The same operator would be responsible for the $2\nu\beta\beta$ decay if it would be OK to treat it in the closure approximation.

$$M_{\rm cl}^{2\nu} \equiv \langle f | \Sigma_{lk} \sigma_l \cdot \sigma_k \tau_l^+ \tau_k^+ | i \rangle,$$

In reality, the closure approximation is not good for the $2\nu\beta\beta$ decay, but we can still consider the corresponding value if we somehow can guess the correct average energy denominator From the correct expression for $M^{2\nu}$

$$M^{2\nu} = \Sigma_m \frac{\langle f || \sigma\tau^+ || m \rangle \langle m || \sigma\tau^+ || i \rangle}{E_m - (M_i + M_f)/2},$$

We can define the radial function $C^{2\nu}{}_{cl}(r)$ the same way as for the genuine $M^{0\nu}$ matrix element, thus

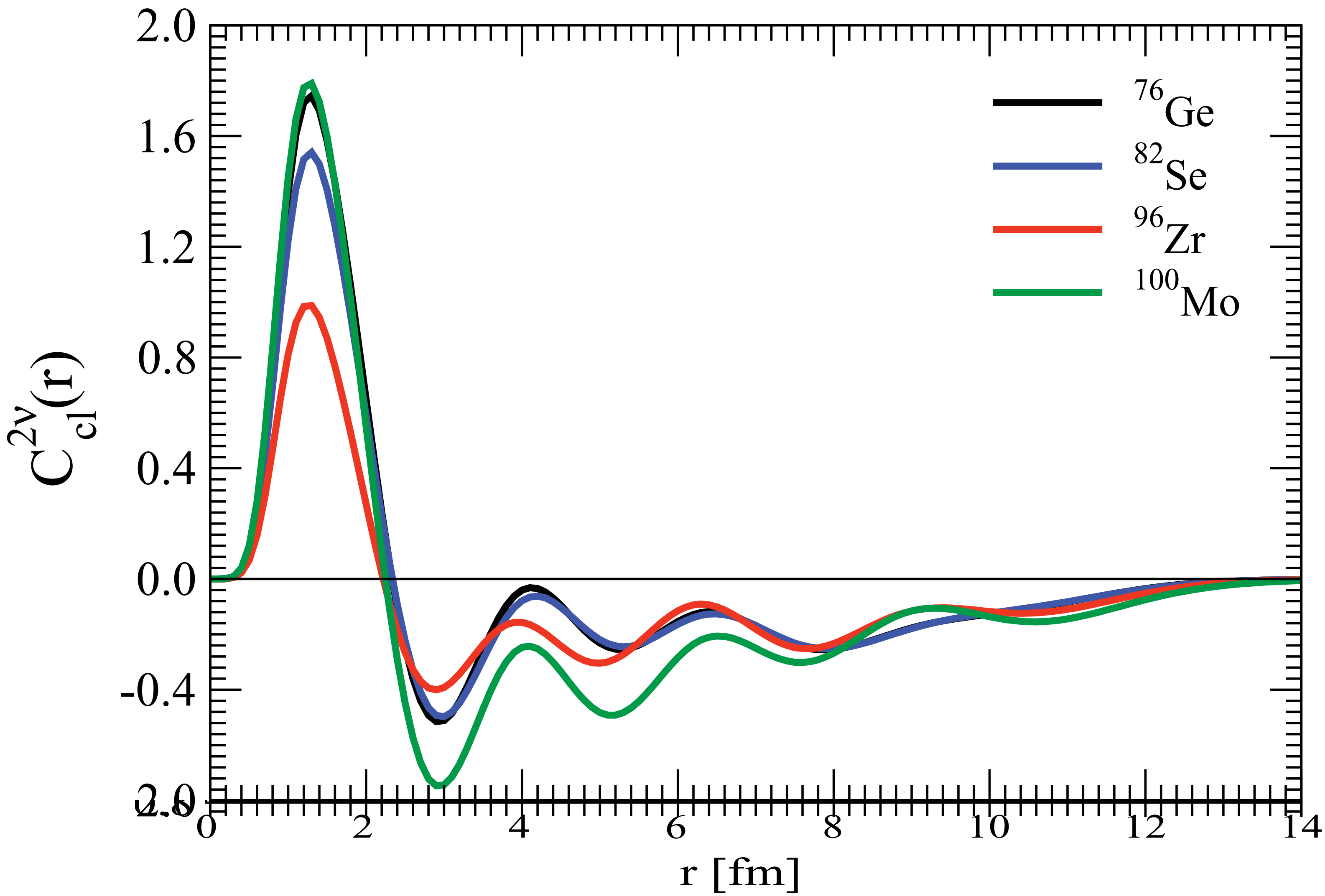
$$C^{2\nu}_{\rm cl}(r) = \langle f | \Sigma_{lk} \sigma_l \cdot \sigma_k \delta(r - r_{lk}) \tau_l^+ \tau_k^+ | i \rangle,$$

$$M^{2\nu}_{\rm cl} = \int_0^\infty C^{2\nu}_{\rm cl}(r)\, dr.$$

It is now clear that, at least formally, the following equality holds: $C^{0\nu}(r) = H(r,E_0)\, C^{2\nu}{}_{cl}(r)$ while $M^{0\nu}_{\rm GT} = \int_0^\infty C^{0\nu}_{\rm GT}(r)\, dr,$

So, if we can somehow determine the function $C^{2\nu}{}_{cl}(r)$ we will be able to determine $C^{0\nu}(r)$ and thus also the ultimate goal, the $M^{0\nu}$.

Functions $C^{2\nu}_{cl}(r)$ evaluated with QRPA for several nuclei. Note that the peak at small r is essentially compensated by the substantial tail at larger r. Besides that the $C^{2\nu}_{cl}(r)$ depends very sensitively on the nuclear parameters used, thus it becomes highly uncertain.



Clearly, determination of $M^{2\nu}_{cl}$ is not easy. We do know the value of $M^{2\nu}$, however $M^{2\nu}_{cl}$ cannot be extracted from the known $2\nu\beta\beta$ decay half-life. That's because while $M^{2\nu}$ and $M^{2\nu}_{cl}$ depend only on the virtual 1^+ states in the intermediate odd-odd nucleus, the weights of individual states are different. Those at higher energies contribute less to $M^{2\nu}$ than to $M^{2\nu}_{cl}$.

This would be OK if the higher energy states have very small both $\langle m| \sigma\tau^+ |i\rangle$ and $\langle m| \sigma\tau^- |f\rangle$. But that is not the case, apparently.

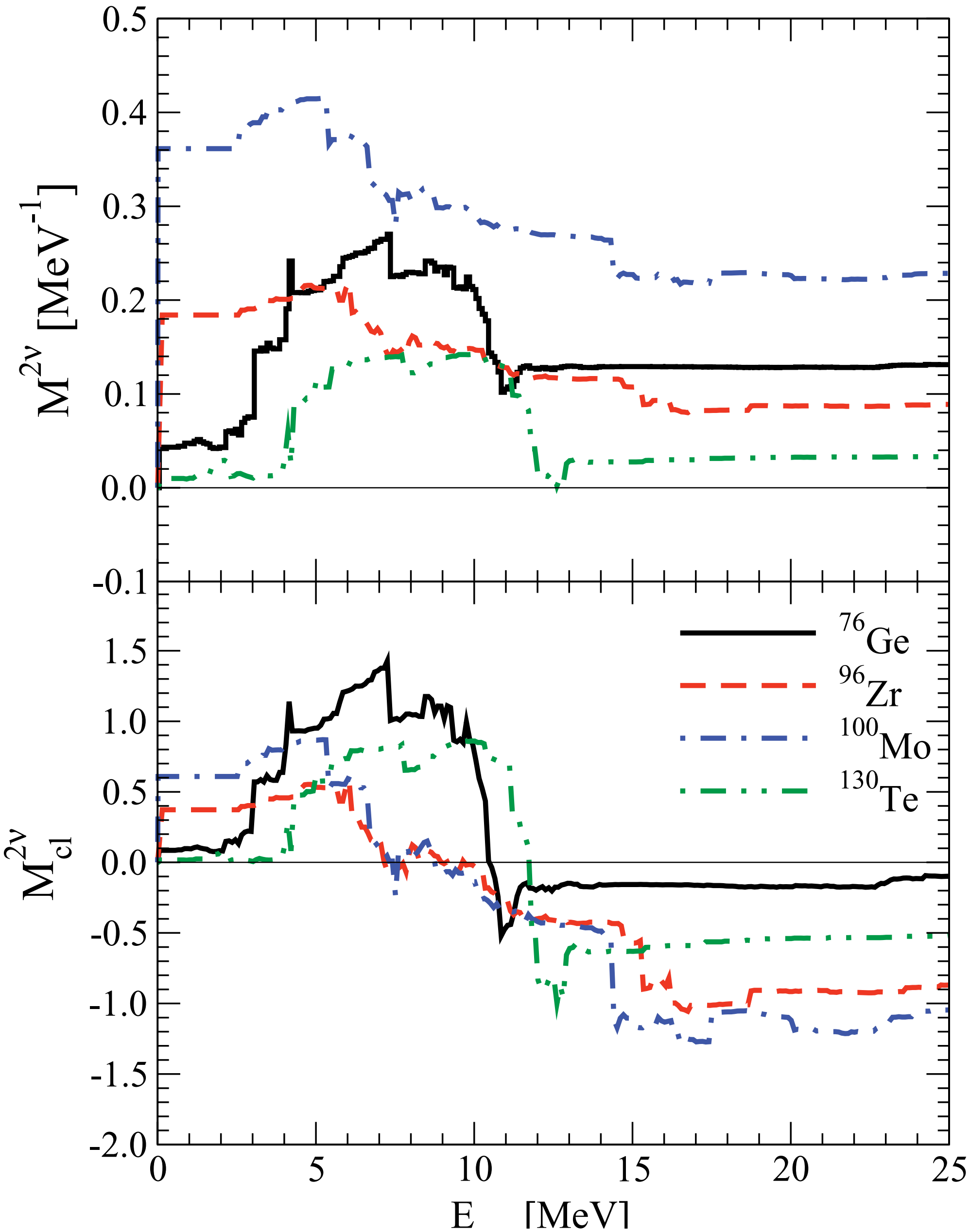
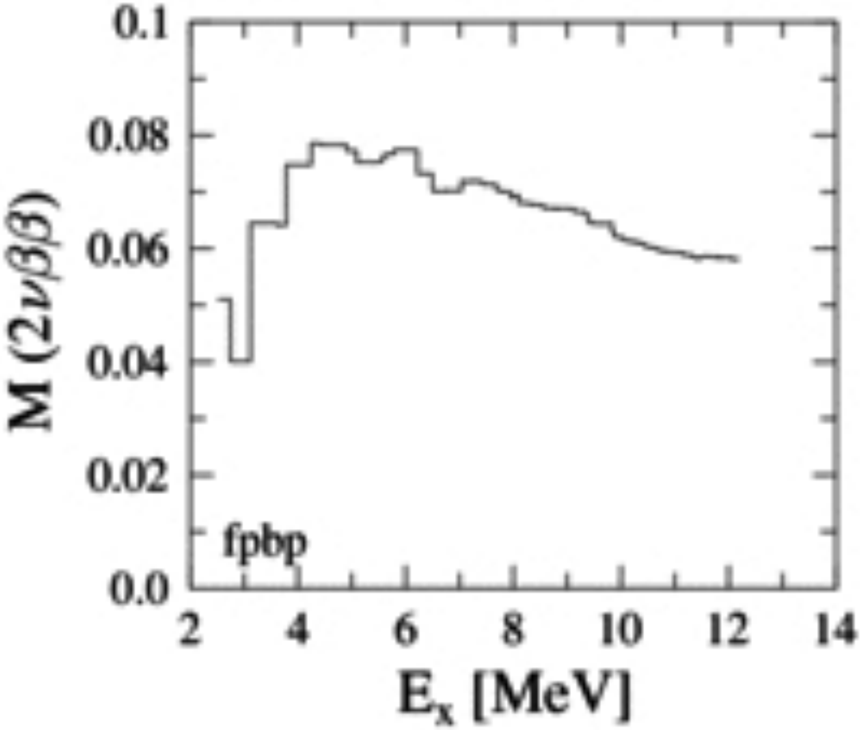
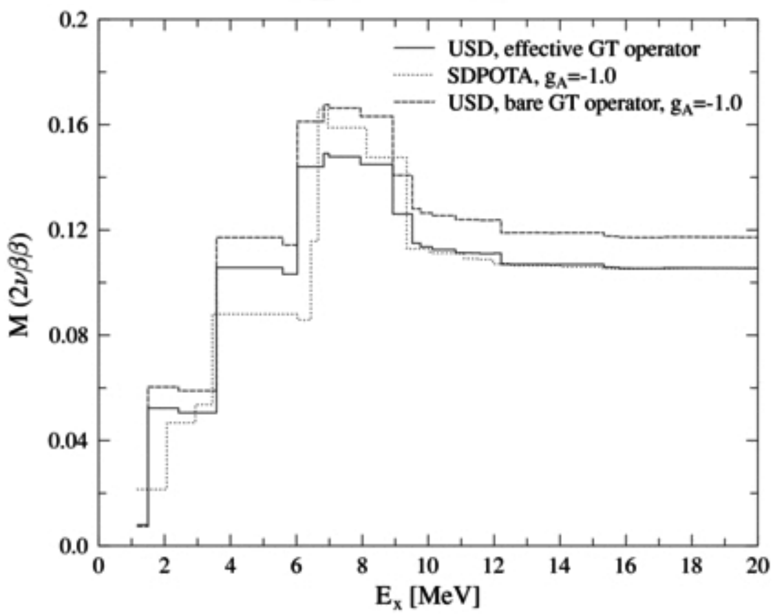


Illustration of the difficulties. In the upper panel are the contributions to the $M^{2\nu}$ from states up to E. Even though the correct value is reached (by design), it is crossed at lower energies, followed by a drop at ~ 10 MeV.

In the lower panel the same calculation is done for $M^{2\nu}{}_{cl}$. In this case the high energy drop is much larger because it is not reduced by the energy denominator present in the true $M^{2\nu}$.

While the states up to ~5 MeV can be studied experimentally, the ~ 10 MeV can not. It is not clear whether they exist or not.





Again, this feature appears to be present in other nuclear models as well. Here are the shell model results for $M^{2\nu}$ in ^{48}Ca (upper panel) and in the model case of ^{36}Ar. (From Kortelainen and Suhonen, J. Phys. *G* **30**, 2003 (2004)).

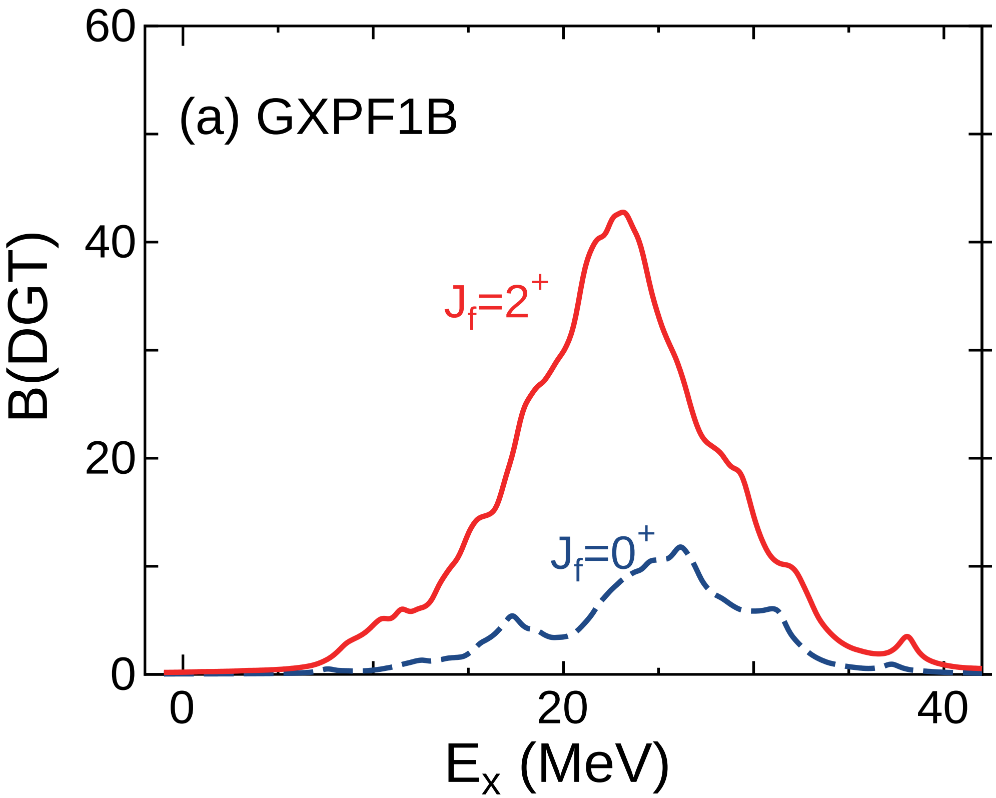
The drop at ~ 10 MeV is again visible, perhaps it is less apparent that in the heavier nuclei treated by QRPA.

Nevertheless, the inherent uncertainty in $M^{2\nu}_{cl}$ is substantial.

However, the $2\nu\beta\beta$ closure matrix element is just one of the states that characterizes the ``double GT" strength function.

$$M^{2\nu}_{\text{cl}} \equiv \langle f | \Sigma_{lk} \sigma_l \cdot \sigma_k \tau_l^+ \tau_k^+ | i \rangle,$$

If we replace <f| by any excited state in the final nucleus, we could trace The distribution of that strength. Here is an example for ^{48}Ca -> ^{48}Ti, evaluated usin the nuclear shell model.

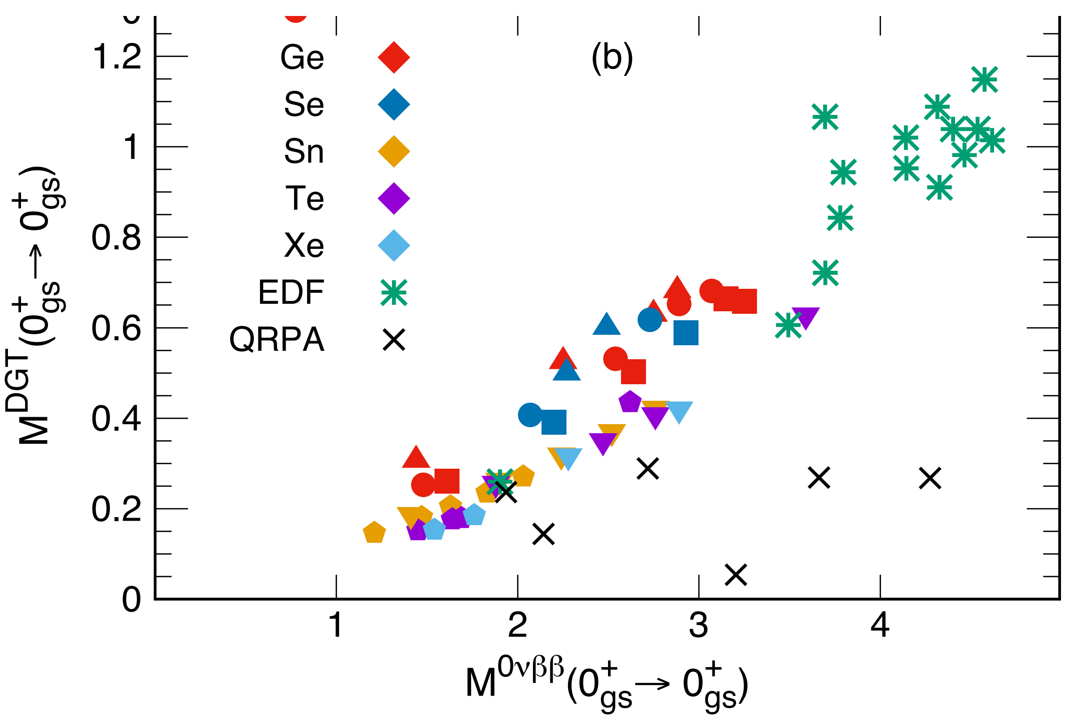


The strength is concentrated in the ``double GT giant resonance" whose central energy depends sensitively on the isovector pairing strenth and its width depends on the isoscalar pairing. This feature could be, perhaps, studied expertimentally by the two nucleon exchange reactions, perhaps something like (*t,p*) or (*p,t*).

Figure from Shimizu et al. 1709.01088

However, what we are really interested in is the ``double GT strength" connecting the ground states of the initial and final nuclei. The calculation predicts that this strength is only ~3×10^{-5} of the total, so its experimental determination is a long shot.

Shimizu et al. suggest (my interpretation) that if the ``giant double-GT resonance" could be observed, its energy and width could be used as the test of the computational procedure. One could then rely on the calculated $M^{2\nu}_{cl}$, which according to them is **proportional** to $M^{0\nu}$.



The proportionality appears to be valid in NSM and EDF. It is not true, however, in QRPA. This is an open problem.

Summary

1) Only small distances, $r < 2$ fm, contribute to the $M^{0\nu}$. That seems to be an universal conclusion, common to all methods where it was tested.
2) That explains, or justifies, why the calculated $M^{0\nu}$ change little with A or Z, unlike $M^{2\nu}$.
3) There is a close relation between $M^{0\nu}$ and the $2\nu\beta\beta$ closure matrix element $M^{2\nu}_{cl}$.
4) If $M^{2\nu}_{cl}$ or, better yet, its radial dependence $C^{2\nu}_{cl}(r)$ could be experimentally determined, it would make the determination of the $M^{0\nu}$ easier.
5) That is not easy. But more work, in theory and experiment, is needed to see how realistic this is.